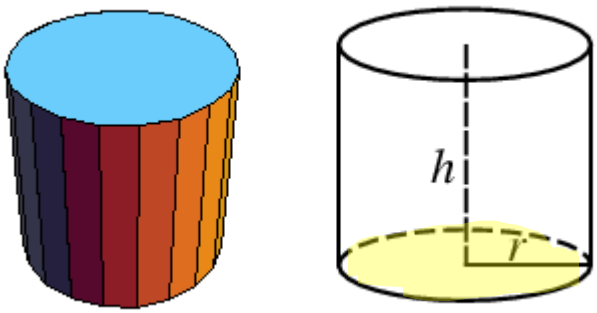
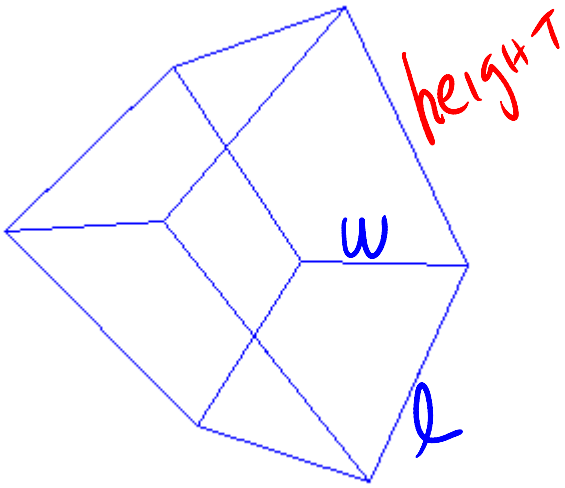
Volume of Known Cross-Sections

Consider the humble cylinder. Use your modeling stuff to make a cylinder.
If you could cut a slice parallel to the base, what would it look like?
How could we find its volume? $V = \pi r^2 h$

From: http://mathworld.wolfram.com/Cylinder.html



What would a slice parallel to the base look like?
How about the volume of a rectangular prism?
From: http://www.shodor.org/interactivate/activities/SurfaceAreaAndVolume



$V = lwh$

We can also think of this as:
Volume = [area of the base] [height]
The base is a rectangle, so we can think of a stack of rectangles.

How about a solid make up of a stack of triangles?
All triangles are the same size [or every "slice" is a congruent triangle]
Make one with your modeling stuff.
How would we find the volume?

V = SOLID (area of one Triangle) height

Can you think of candy whose cross sections are circles, rectangles, or triangles?

Now with your modeling stuff make a solid whose cross sections are all circles but all of the circles have different radii. [from very, very small to larger]
What does it look like? Carrot

or parsnip
or Cone

Make a solid whose cross sections are squares – but each square has a different side length.
What does it look like? square pyramid

Images from Google Images [What are the cross sections?]

circles

Square

Luxor Hotel [Las Vegas]

Our Handy-Dandy Formula to find the Volume of a Solid with Known Cross-Sections ☺ ☺ ☺ ☺

Volume = [area of a cross-section]*[height]

We need to figure out how to use calculus for this! Since we are "*summing up*" the *areas* of each cross-section, then should we integrate or differentiate?

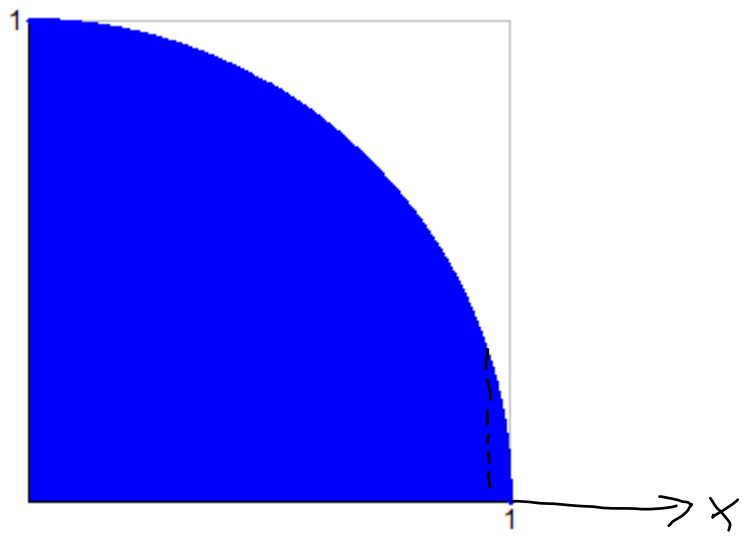
Our Calculus formula: ☆☆ ☆☆

$V = \int_a^b A(x)\, dx$ where $A(x)$ is the area of a cross section

Let's consider the following:

NOTE: Cross-section to the X-AXIS

Find the volume of the solid created on a region whose base is the function $y=\sqrt{1-x^2}$ for $0 \le x \le 1$. For this solid, each cross section perpendicular to the $x-$axis is a square. Let's try to figure out what this looks like.



http://mathdemos.org/mathdemos/sectionmethod/sqrcross75.gif
http://mathdemos.org/mathdemos/sectionmethod/sqrcross75slab.gif

The volume can be found with our handy-dandy formula!

$\int_0^1 A(x)\ dx$ where $A(x)$ is the area of a cross section.

Each cross section is a square so the area of a cross section is equal to s^2 where $s=\sqrt{1-x^2}$ and our side lengths will vary according to our upper and lower bounds

$a(x) = s^2$

$\int_0^1 \left[\sqrt{1-x^2}\right]^2 dx$

$\int_0^1 A(x)\,dx$

$a(x) = s^2$
$= \left(\sqrt{1-x^2}\right)^2$

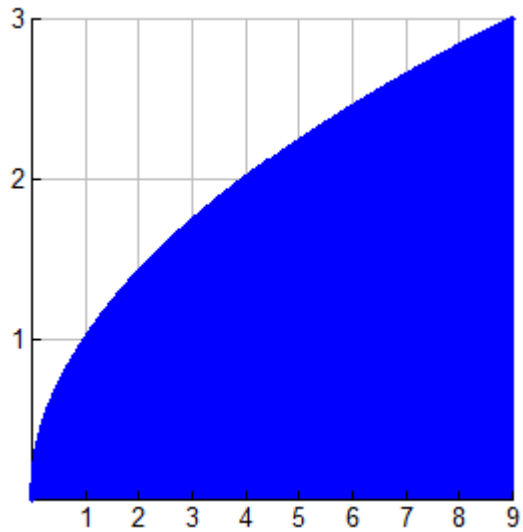
$$V = \int_0^1 (1-x^2)\,dx$$

$$= x - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{2}{3} \text{ c.u.}$$

Now let's find the volume of the solid formed with a base function $y = \sqrt{x}$ with $0 \le x \le 9$, but with semi-circles as our cross sections that are perpendicular to the $x-$axis

Here is our base:



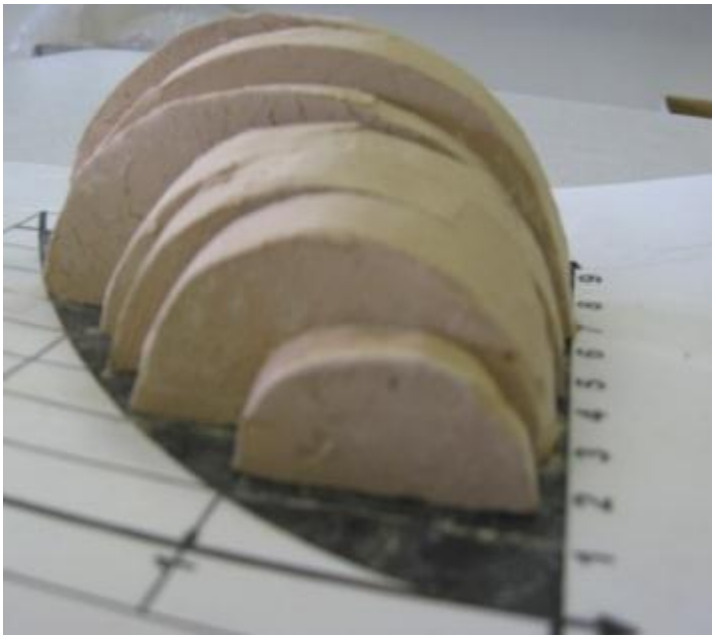
This one is hard to visualize.

Here is an awesome video:

http://mathdemos.org/mathdemos/sectionmethod/sqrtcirccross75.gif

http://mathdemos.org/mathdemos/sectionmethod/sqrtcirccross75slab.gif

And here is an image of one:



Once again, $V = \int_0^9 A(x)\ dx$ $\qquad A(x) = \frac{\pi}{2} r^2$

$$V = \int_0^9 \frac{\pi}{2}\left(\frac{1}{2}\sqrt{x}\right)^2 dx \qquad r = \frac{1}{2}\sqrt{x}$$

$$A(x) = \frac{\pi}{2}\left(\frac{1}{2}\sqrt{x}\right)^2$$

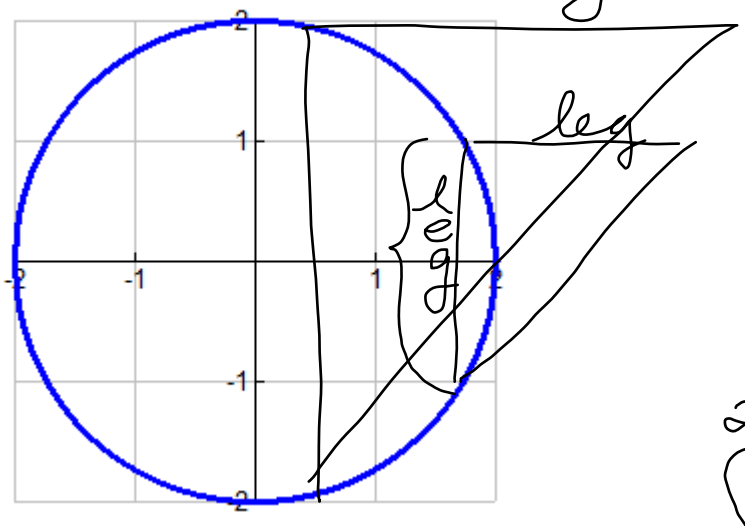
$$= \int_0^9 \frac{\pi}{2}\left(\frac{x}{4}\right) dx$$

$$= \frac{\pi}{8}\int_0^9 x\ dx \qquad = \frac{\pi}{8}\left(\frac{x^2}{2}\right)\Big|_0^9$$

One more, a circular base [a radius of 2]
with right isosceles triangle whose leg is on the base
and each cross section is perpendicular to the $x-$axis
http://mathdemos.org/mathdemos/sectionmethod/circisoscross75slab.gif

$$y = \pm\sqrt{4 - x^2}$$

$y = \pm\sqrt{4-x^2}$



$leg = 2\sqrt{4-x^2}$

$a(x) = \frac{1}{2}b^2$

b = leg length

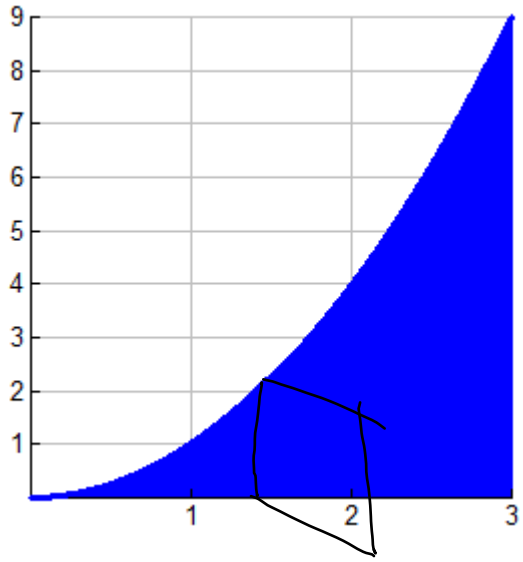
Area of a cross-section is $\int_{-2}^{2} a(x)\,dx$

$= \int_{-2}^{2} \frac{1}{2}\left[2\sqrt{4-x^2}\right]^2 dx$

You try to set up the following:

Find the volume of the solid created on a region whose base is the function $y = x^2$ for $0 \le x \le 3$

And the cross-sections are squares, semi-circles, and right isosceles triangles with the leg on the base and each cross section is perpendicular to the x–axis



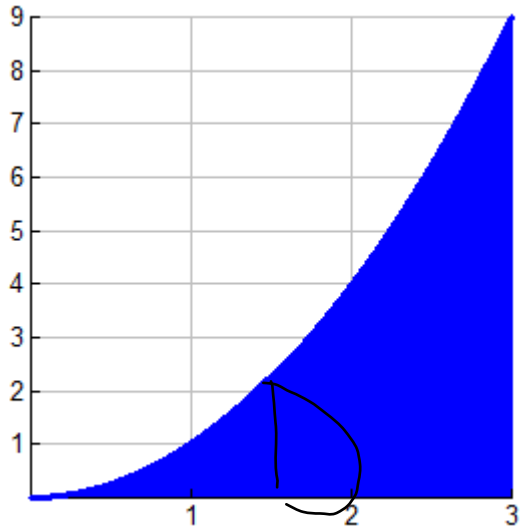
$$\int_0^3 a(x)\,dx$$

$$= \int_0^3 (x^2)^2\,dx$$

squares

$a(x) = s^2$

$s = x^2$



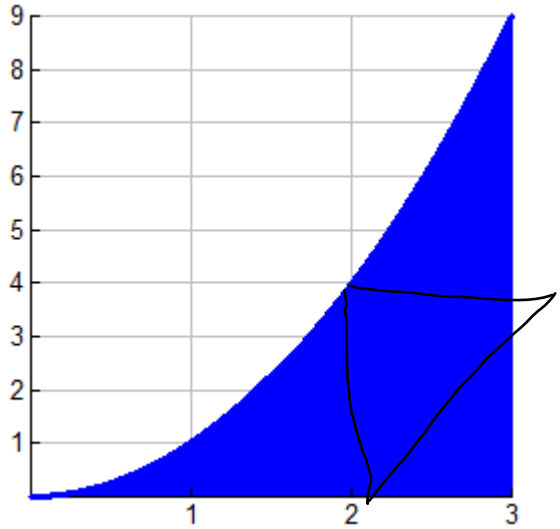
$$\int_0^3 a(x)\,dx$$

$$= \int_0^3 \frac{\pi}{2}\left(\frac{x^2}{2}\right)^2 dx$$

semi-circles

$a(x) = \frac{\pi}{2} r^2$

$r = \frac{x^2}{2}$



$$V = \int_0^3 a(x)\,dx$$

$$V = \int_0^3 \frac{1}{2}(x^2)^2\,dx$$

RIGHT-ISOSCELES Δ

$a(x) = \frac{1}{2} b^2$

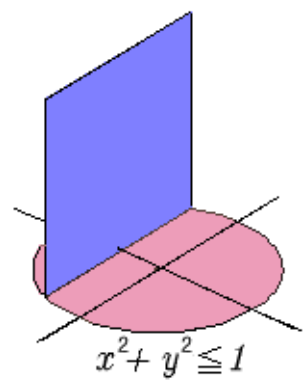
$b = x^2$

Now let's consider the relation $x^2 + y^2 = 1$
Find the volume of an object whose base is the relation above and when the cross sections are squares and each cross section is perpendicular to the $x-$axis

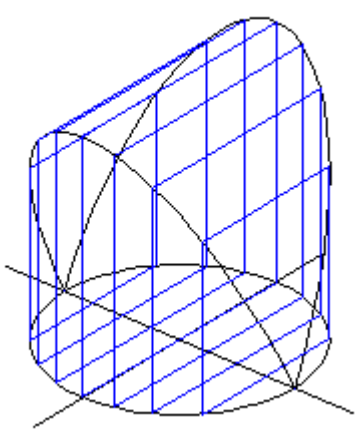
Here is a cool website where we can visualize this
http://www.ies.co.jp/math/java/samples/renshi.html

There is a solid whoese bottom face is the circle $x^2 + y^2 \leqq 1$.
And every cross-section of the solid perpendicular to x-axis is a square.

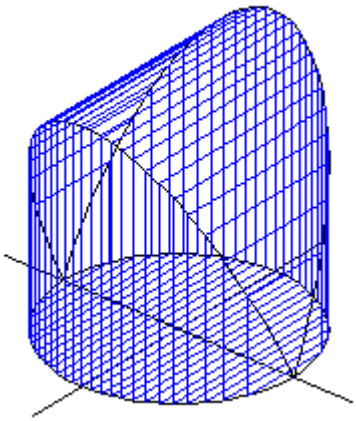
Find the volume of the solid.



Here's what it looks like with a few squares:

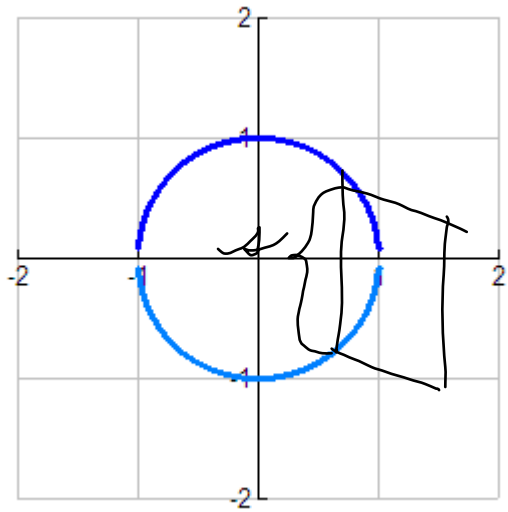


With a lot more squares:



Cardboard Model

Area of a cross section = area of a square
Here is the base of the solid.



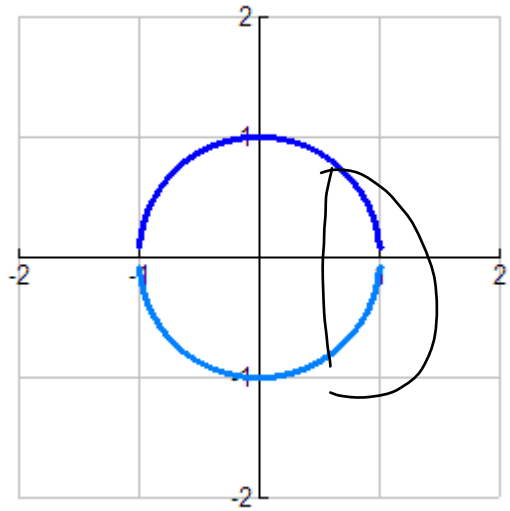
If $x^2 + y^2 = 1$, then $y = \pm\sqrt{1 - x^2}$ with a domain of [-1,1]

$$A(x) = s^2$$

$$s = 2\sqrt{1-x^2}$$

$$V = \int_{-1}^{1} \left[2\sqrt{1-x^2}\right]^2 dx$$

Let's change one thing! What would be the volume if the cross sections were *semi-circles* instead of squares? And, each cross section is perpendicular to the x–axis



$$a(x) = \frac{\pi}{2} r^2$$

$$r = \sqrt{1-x^2}$$

$$V = \int_{-1}^{1} \frac{\pi}{2}\left(\sqrt{1-x^2}\right)^2 dx$$

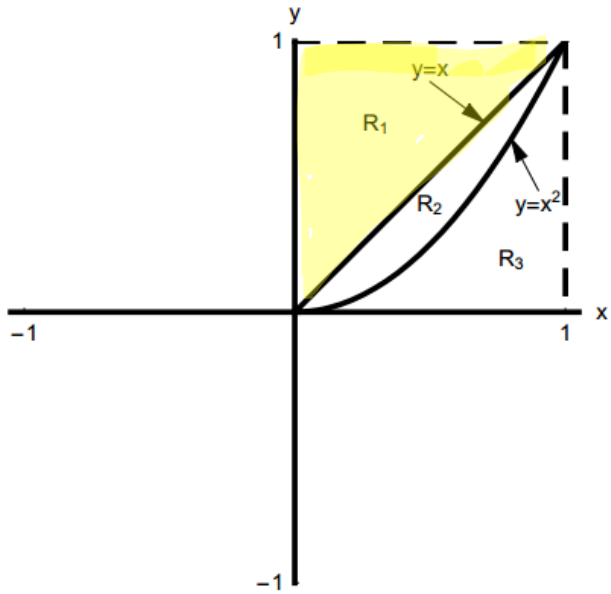
Lots of cool photos at:

http://mathdemos.gcsu.edu/mathdemos/sectionmethod/sectiongallery.html

Homework

Handout [see next pages if you were not in class]

Mr. Zab's Volume of Known Cross Sections Handout [Day One]



Let R_1 be the region in the first quadrant bounded by y = x, the y-axis, and the line y = 1. Let R_2 be the region in the first quadrant bounded by $y = x^2$ and y = x. Let R_3 be the region in the first quadrant bounded by $y = x^2$, the x-axis, and the line x = 1.

Instructions:
State *A(x)* [the area of a cross section]
Set up the integral to find the volume
Use your TI to calculate the volume

16. Let R_1 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are squares, find the volume of the solid.

17. Let R_1 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are semicircles, find the volume of the solid.

18. Let R_2 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are squares, find the volume of the solid.

19. Let R_2 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are semicircles, find the volume of the solid.

20. Let R_3 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are squares, find the volume of the solid.

21. Let R_3 be the base of a solid in the x-y plane. If cross sections of the solid perpendicular to the x-axis are semicircles, find the volume of the solid.