Student Study Session

Justifications

http://www.zendog.org/homework

Some general tips

1. Write sentences that use Calculus but do not just state a rule

Here is an example of a solution that would not be sufficient:

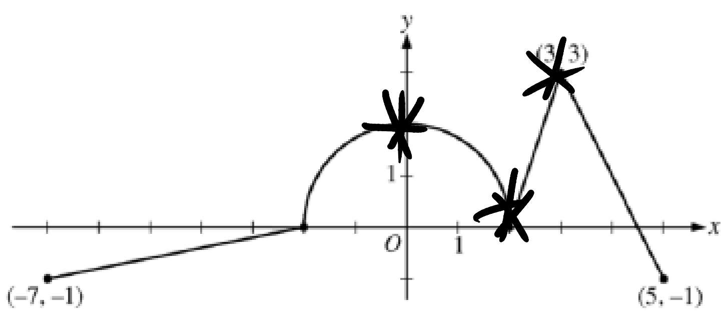
The graph of f has relative extrema at $x = 3\ and\ x = 5$ because f' changes either from negative to positive or positive to negative.

You must be specific. Here is a better way:

The graph of f has a relative maximum at $x = 3$ because f' changes from positive to negative at $x = 3$. The graph of f has a relative minimum at $x = 5$ because f' changes from negative to positive at $x = 5$.

2. Use what is given you – if given a graph, then you are expected to use that graph in some way

Example: #2 on handout



Graph of g'

Find the x –coordinate of each point of inflection on the graph of $y = g(x)$ on the interval $-7 < x < 5$

Since we are given the graph of g' you should probably use it.

Why would this answer not receive any points?

The graph of g has a point of inflection at $x = 0$ because it changes from increasing to decreasing at $x = 0$.

How can we make this a better solution?

at $x = 0$ and $x = 3$ the graph of g' changes from increasing to decreasing. at $x = 2$ the graph of g' changes from decreasing to increasing, so) g has POI at $x=0, x=2, x=3$

3. Use standard mathematical notation [You will probably save some time if you do.]

There is no need to write out phrases like "the second derivative of f is positive on the open interval 3 to five" when you can simply write $f''(x) > 0$ on $3 < x < 5$

4. No reader will assume that you know what you are talking about. So avoid being general.

Not sufficient:

The graph is increasing or the slope is positive or the function is differentiable.

What graph? The slope of what? What function?

5. Use the function(s) given.

If given the function, $v(t)$, then don't be talking about $f(x)$. No one will assume that you are referring to the velocity, especially if you need to be finding acceleration or position.

6. You cannot justify an incorrect answer.

Example:

You see that f' changes from negative to positive values at $x = 3$ and you say this:

The graph of f has a relative maximum at $x = 3$ because f' changes from negative to positive values at $x = 3$.

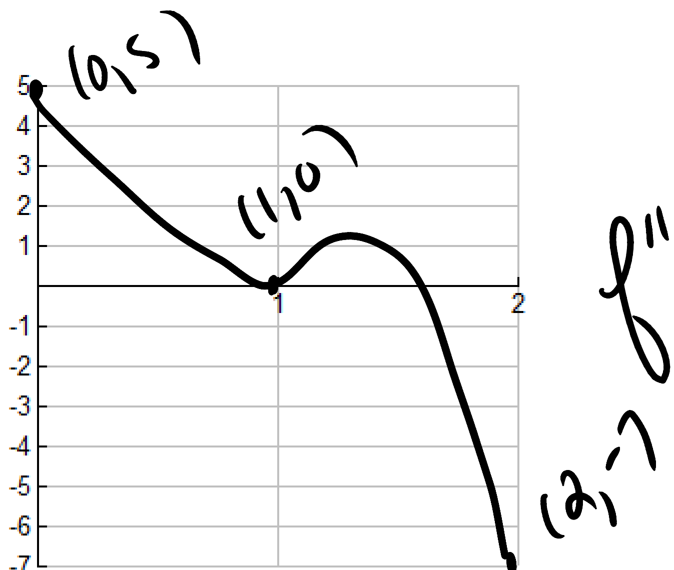
7. Sign charts do not receive any points. AP readers are instructed to ignore any sign charts and only read written out justifications. [You can make a sign chart but you must justify with sentences.]

8. When reading a table of values, do not assume you know what is happening between the selected values unless given specific information. Notice the difference between these two set-ups:

(a) The polynomial function f has selected values for its second derivative f'' given in the table below.

x	0	1	2	3
$f''(x)$	5	0	-7	4

In this case, we can justify that the graph of f has a point of inflection somewhere on the interval $0 < x < 2$. [You may NOT assume that the only zero is at $x = 1$. $f''(-0.1)$ could also be equal to zero.]



No P of I at x = 1

(b) The function f has selected values for its second derivative, f'', given in the table below. f'' is a <u>strictly decreasing function</u> for the interval [0, 3].

x	0	1	2	3
$f''(x)$	5	0	-3	-4

Because we are told that f'' is a strictly decreasing function for the interval [0, 3], then we may justify that at $x = 1$, the graph of f has a point of inflection.

9. If a function is continuous, then the following theorems can be applied:

Intermediate Value Theorem

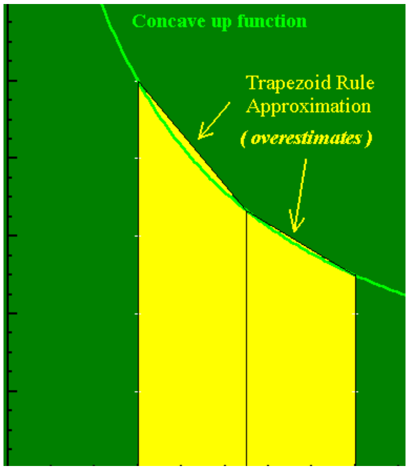
Extreme Value Theorem

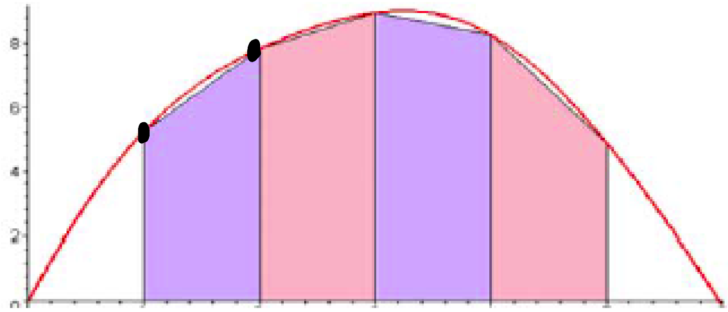
Fundamental Theorem of Calculus

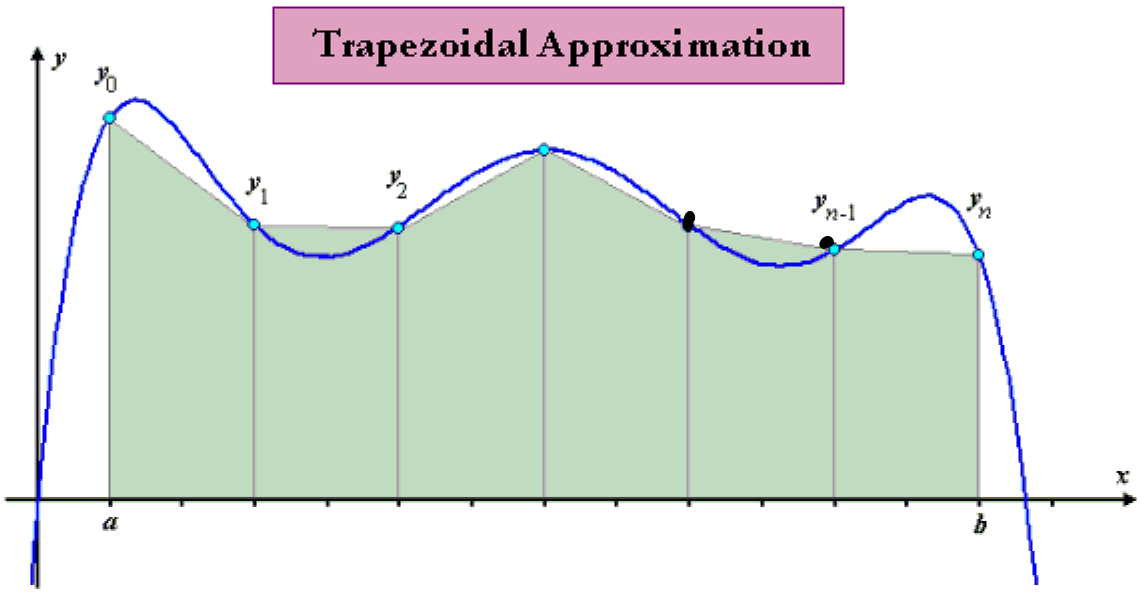
If a function is differentiable, then all of the above can be applied AND the Mean Value Theorem can also be applied. [Be sure to mention the necessary conditions in your justification.]

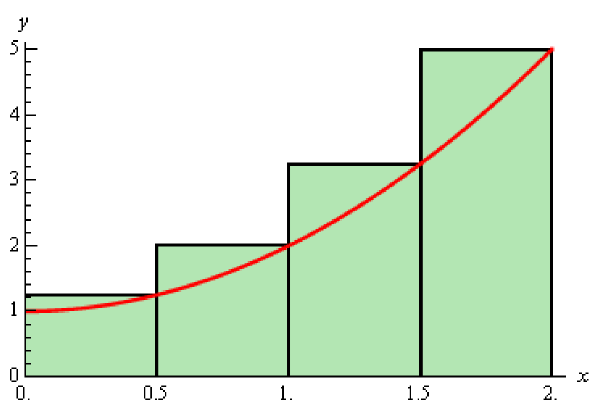
10. Quit while you are ahead. You don't need to "fill the white space". You risk contradicting yourself or making false statements which will deprive you of points.

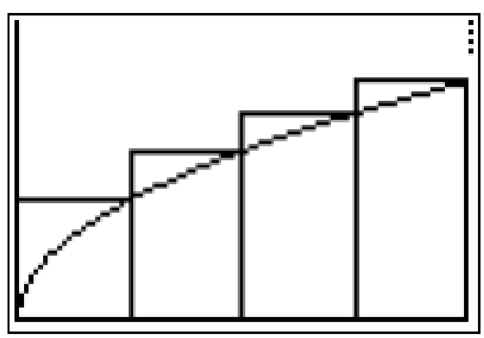
Let's ponder the "over" or "under" scenario. If you are stuck, then just draw a picture.



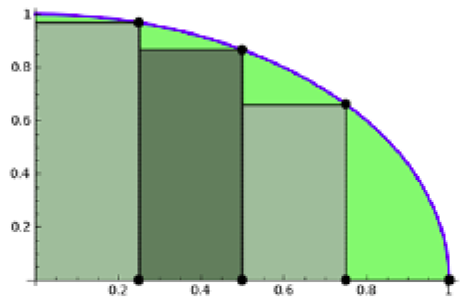




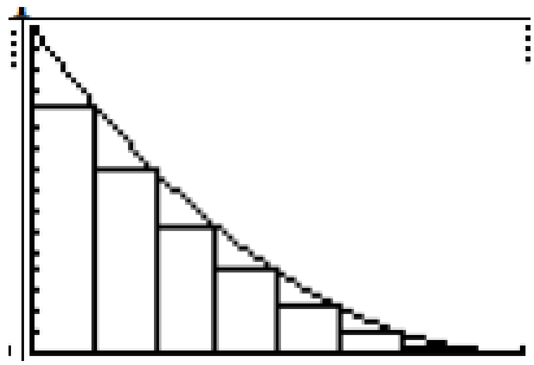




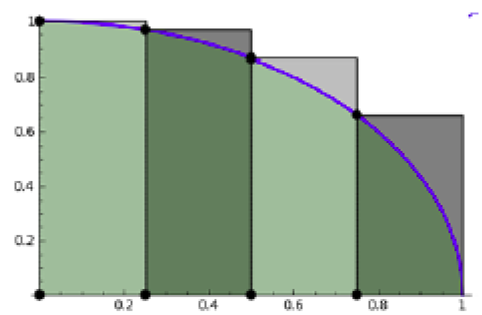
Right Riemann Sums [increasing]

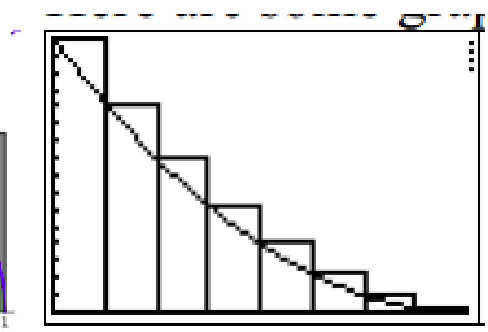


Right Riemann Sum

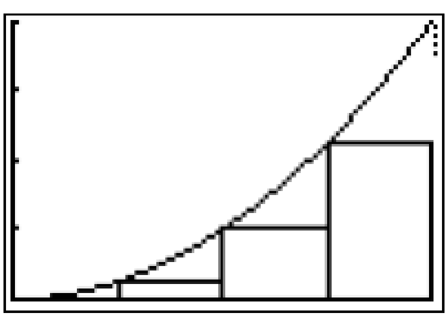


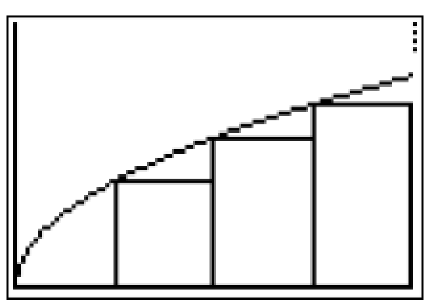
Right Riemann Sums [decreasing]





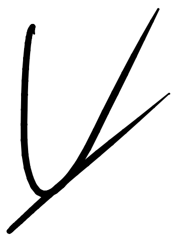
Left Riemann Sums [decreasing]

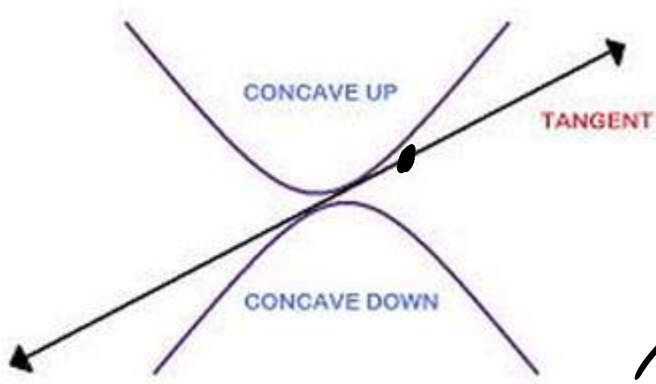


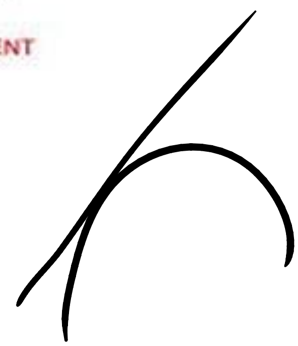


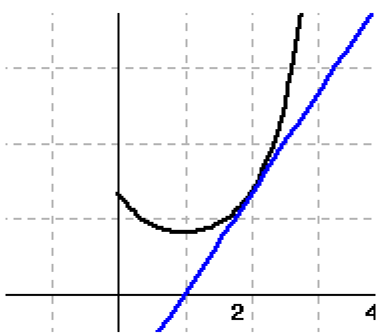
Left Riemann Sums [increasing]

Tangent Line Approximations

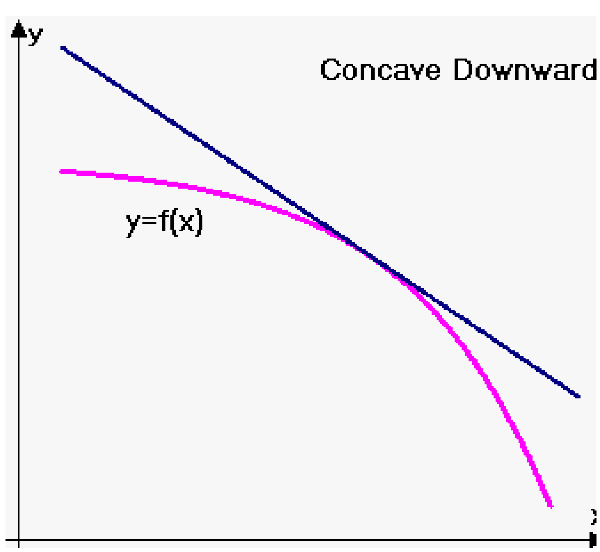








Concave up



#1 The "over or under" explanation

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1+3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

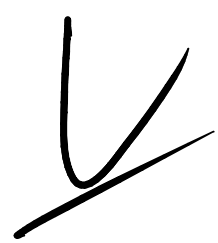
(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

An equation of the tangent line is
$y = 2 + 8(x-1)$.

(b)

$f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval
$1 \le x \le 1.1$,
$\frac{d^2y}{dx^2} = y^3(1+3x^2y^2) > 0$ on this interval. concave up



Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

A common question on the free response [but occasionally in the multiple-choice] is to find the meaning of an integral or a derivative. If you are uncertain, then consider thinking about the solution's units.

Example #4

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table.

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

H(t) height of TRAP OR RECT

$$\frac{1}{10}\int_0^{10} H(t)\,\underline{dt}$$

°C

dividing by minutes

°C*MIN

dt width in MIN

gives us AV. TEMP, IN °C, DURING the INTERVAL $0 \le t \le 10$ MIN

(c) $\int_0^{10} H'(t)dt$

$= H(t) \Big|_0^{10}$

$= H(10) - H(0)$

$= 43°C - 66°C$

Tells us that the TEA COOLED by 23°C DURING $0 \le t \le 10$ MINUTES

Justifying with graphs

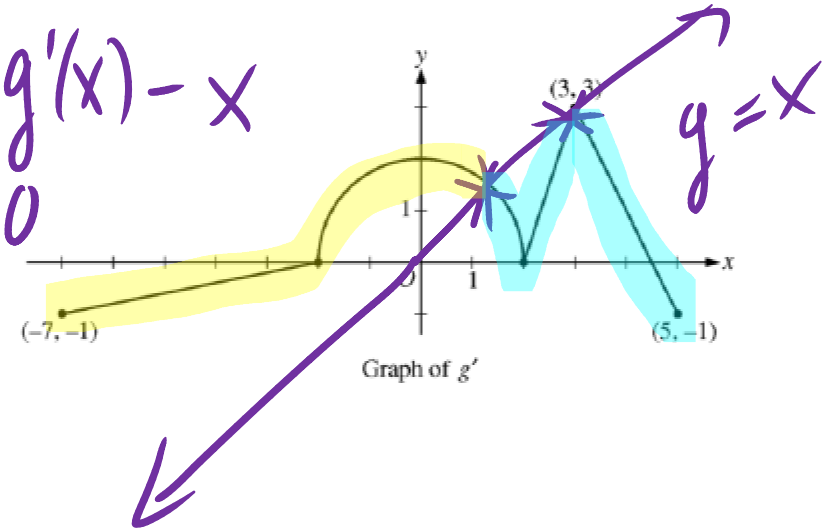
In our handout, graphs are used in #2, #5, #6, #7, and #8. The graph of a derivative is used each time but in different scenarios.

The first thing you should do is ask yourself, "What do I see?"

$h'(x) = g'(x) - x$

$h'(x) > 0$

$g'(x) > x$



Graph of g'

C.V.

$g'(x) = x$

$x = 3$

$x = \sqrt{2}$

(c) This is a tricky question! Don't' skip steps! [It's okay to draw on your graph]

$$h(x) = g(x) - \frac{1}{2}x^2$$

$h'(x) = g'(x) - x$

$0 = g'(x) - x$

$x = g'(x)$

Semi-circle

$y = \sqrt{4 - x^2}$

$x = \sqrt{4 - x^2}$

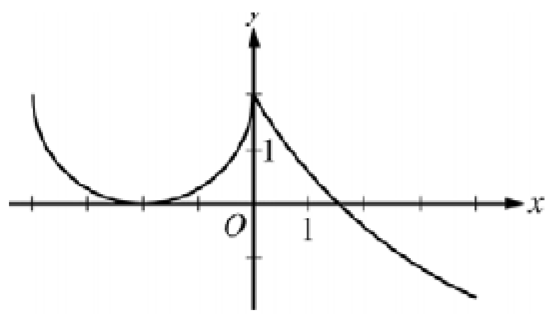
$x^2 = 4 - x^2$

$2x^2 = 4$

$x^2 = 2$

$h'(x) = g'(x) - x > 0$ on $-7 < x < \sqrt{2}$
$h'(x) = g'(x) - x < 0$ on $\sqrt{2} < x < 3$ and $3 < x < 5$
at $x = \sqrt{2}$ h has a rel max because
h' changes from positive to negative
at $x = 3$ h has neither a rel min
nor a rel max

#5



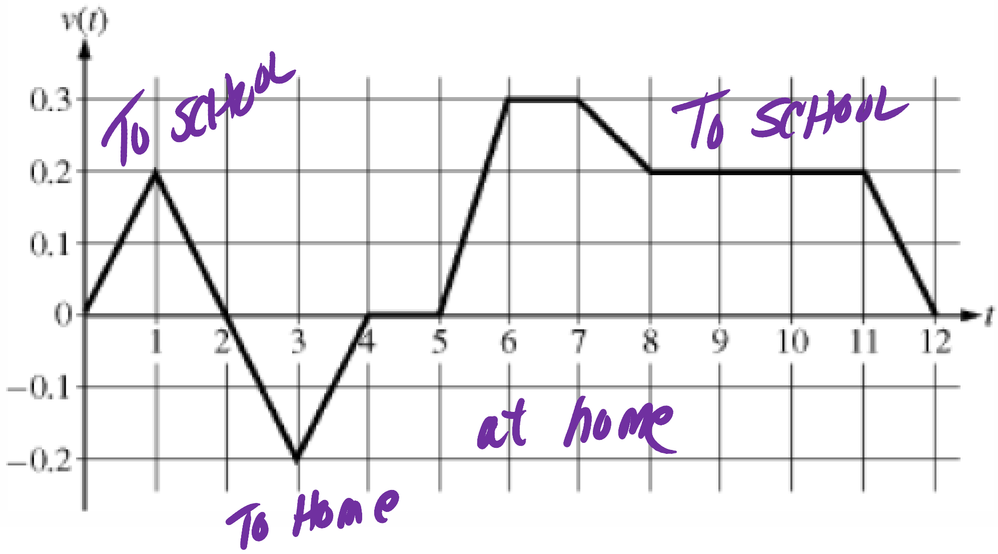
Graph of f'

What do you see?

(a)

(c)

#6 and #8 are similar



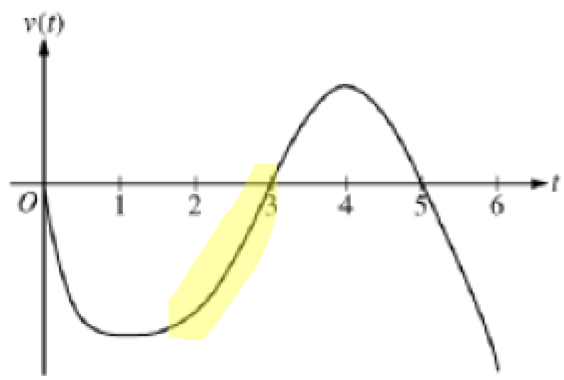
What do you see?

(a) The meaning of $\int_0^{12} |v(t)|\, dt$

TOTAL DISTANCE TRAVELED
IN MILES DURING $0 \le t \le 12$ MINUTES

(b) Give a reason for your answer.

#6 The speed question



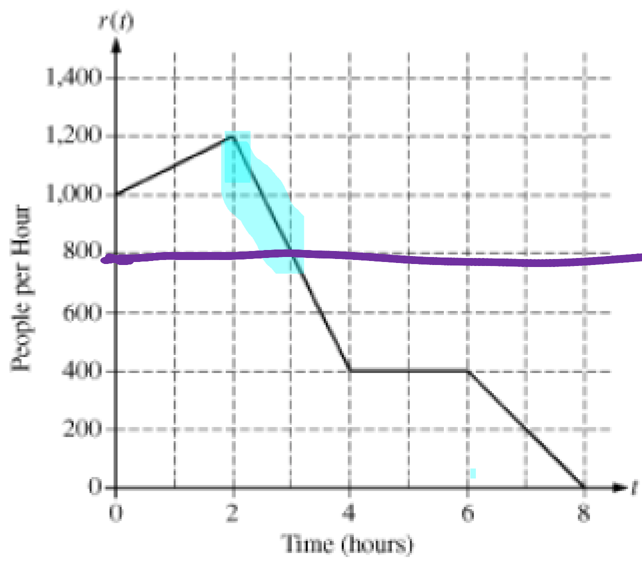
Graph of v

(c) Last year's exam ask about speed more than once.

(b) Using a theorem.

(d) Connecting acceleration to velocity

#7



$\frac{800 \text{ People}}{\text{hr}}$

RATE AT WHICH PEOPLE LEAVE THE LINE

(b) It's still okay to draw on your graph

The RATE AT WHICH people enter is greater than the RATE AT WHICH leave the line because $r(t) > 800$ on $2 < t < 3$ hence the number of people in line is increasing on $2 < t < 3$

Some problems require a lot of work for the explanations

#3

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.

Here's what you can't say:

$L'(t)$ must equal zero at least 3 times

because must equal zero whenever $L(t)$ changes from increasing to decreasing, or ~~fron~~ decreasing to increasing, which we can only be sure of happening (based on the chart) between $t=3$ and $t=4$; between $t=4$ and $t=7$; and between $t=7$ and $t=8$.

3 times is the correct solution but since the student assumed that L(t) changes from increasing to decreasing on the interval 3< t <4

How to earn all of the points:

You could use a theorem. You should mention that L(t) is a differentiable function [which makes $L'(t)$ continuous] Another approach would be to use the Extreme Value Theorem but this takes more time.

How many points do you think this student received?

2 times

Here is a sample of a complete solution:

L **is differentiable on [0, 9] so the Mean Value Theorem guarantees that** $\boldsymbol{L'(t)}$ **must take on the following average rates of change:**

$$\frac{L(3)-L(}{3-0} = \frac{176-120}{3} > 0$$

$$\frac{L(4)-L(3)}{4-3} = \frac{126-156}{1} < 0$$

$$\frac{L(7)-\ (4)}{7-4} = \frac{150-126}{4} > 0$$

3

$$\frac{L(9)-L(7)}{9-7} = \frac{0-150}{2} < 0$$

So, the values of $L'(t)$ must change from positive to negative, negative to positive, and positive to negative, so the IVT guarantees that $L'(t) = 0$ *at* least three times in the interval [0, 9]

Written for the Colorado Legacy Foundation Student Study Session

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