#13 → change to g(x)

AP Calculus
2.4 Worksheet

Cofield

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is a difference quotient? $\frac{f(a+h)-f(a)}{h}$

2. How do you find the slope of a curve (aka slope of the tangent line to a curve) when $x = a$?

$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ * must have "$\lim_{h\to 0}$" or its not slope of curve at $x=a$.

3. What is a *normal line* ?

A normal line is the line perpendicular to a tangent line.

4. What is the difference between the AVERAGE RATE OF CHANGE and INSTANTANEOUS RATE OF CHANGE?

Avg Rate of Change = slope of secant line (2pts) $m = \frac{y_2-y_1}{x_2-x_1}$ | Instantaneous Rate of Change = Slope of curve at a point $= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

5. Find the average rate of change of each function over the indicated interval.

a) $h(x) = e^x$ on $[-2, 0]$

avg. rate of chg $= \frac{h(0)-h(2)}{0-(-2)}$

$= \frac{e^0 - e^{-2}}{2}$

$= \boxed{\frac{1-e^{-2}}{2}}$

b) $k(x) = 2 + \sin x$ on $[-\pi/2, \pi/2]$

avg. rate of chg $= \frac{k(\frac{\pi}{2}) - k(-\frac{\pi}{2})}{\frac{\pi}{2} - (-\frac{\pi}{2})}$

$= \frac{2+\sin(\frac{\pi}{2}) - (2+\sin(-\frac{\pi}{2}))}{\pi}$

$= \frac{2+1-(2-1)}{\pi}$

$= \boxed{\frac{2}{\pi}}$

c) $f(x) = x^2 - x$ on $[1, 3]$

avg rate of chg $= \frac{f(3)-f(1)}{3-1}$

$= \frac{(3)^2-(3)-((1)^2-(1))}{2}$

$= \frac{6-(0)}{2}$

$= \boxed{3}$

6. Let $f(x) = x^3$.

a) Write and simplify an expression for $f(a+h)$.

$f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$

I used pascal's Δ for coefficients...

1
1 1
1 2 1
1 3 3 1

b) Find the slope of the curve at $x = a$.

$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\cancel{a^3}+3a^2h+3ah^2+h^3-\cancel{a^3}}{h} = \lim_{h\to 0} \frac{h(3a^2+3ah+h^2)}{h}$

$= \lim_{h\to 0} (3a^2+3ah+h^2) = 3a^2 + 3a(0) + (0)^2 = \boxed{3a^2}$ ← "formula" for slope at any value a.

c) When does the slope equal 12?

Slope = 12
$3a^2 = 12$
$a^2 = 4$
$a = \pm 2$

Since x=a, then the slope of f(x) is 12 when $x = \pm 2$.

d) Write the equation of the tangent line to the curve at $x = 4$ $y - 64 = 48(x-4)$

① Need pt $f(4) = 64$ ② Need slope at $x=4$ $3(4)^2 = 48$

e) Write the equation of the normal line to the curve at $x = 4$ $y - 64 = -\frac{1}{48}(x-4)$

Need ⊥ slope at $x=4$ ... $-\frac{1}{48}$

7. Let $g(x)=\sqrt{x}$

a) Find the $\underbrace{\text{average rate of change}}$ from $x=4$ to $x=9$. ARDC $= \frac{g(9)-g(4)}{9-4} = \frac{\sqrt{9}-\sqrt{4}}{5} = \frac{1}{5}$

ARDC = slope btwn 2 pts

b) Find the $\underbrace{\text{instantaneous rate of change}}$ at $x=9$.

IROC = limit of AROC

direct subst gives $\frac{0}{0}$ ... Do something else!

$$\lim_{h\to 0} \frac{g(9+h)-g(9)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-\sqrt{9}}{h} = \lim_{h\to 0} \frac{(\sqrt{9+h}-3)}{h} \cdot \frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{\sqrt{9+0}+3} = \boxed{\frac{1}{6}}$$

* You can also find $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$ first and then evaluate your answer when $x=9$. like we did on the notes.

c) Write the equation of the tangent line when $x=9$

① Need pt $g(9)=3$ $(9,3)$

② Need slope at $x=9$ $m=\frac{1}{6}$

③ Write line... $y-3=\frac{1}{6}(x-9)$

d) Write the equation of the normal line when $x=9$.

$m_\perp=-6$ $y-3=-6(x-9)$

8. Let $y=\frac{1}{x-1}$. Find the slope of the curve at $x=2$. Using this slope, write the equation of the tangent line and the equation of the normal line at that point. IROC

Need common denom.

$$\text{slope @}x=2 = \lim_{h\to 0} \frac{y(2+h)-y(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(2+h)-1}-\frac{1}{2-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h}-1\cdot\frac{(1+h)}{(1+h)}}{h} = \lim_{h\to 0} \frac{\frac{1-(1+h)}{1+h}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{-h}{1+h} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{1+h} = \frac{-1}{1+0} = -1$$ ... slope of y at $x=2$

point: $x=2$ $y=\frac{1}{2-1}=1$

Tangent Line: $y-1=-1(x-2)$

Normal Line: $y-1=1(x-2)$

9. Let $y=x^2-3x-2$. Find the slope of the curve at $x=0$. Using this slope, write the equation of the tangent line and the equation of the normal line at that point. IROC

$$\text{slope ... @}x=0 \quad \lim_{h\to 0} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2-3(0+h)-2-(0^2-3(0)-2)}{h} = \lim_{h\to 0} \frac{h^2-3h-2+2}{h}$$

$$= \lim_{h\to 0} \frac{h^2-3h}{h} = \lim_{h\to 0} \frac{h(h-3)}{h} = \lim_{h\to 0} h-3 = 0-3=-3$$ ... slope @ $x=0$

point: $x=0$ $y(0)=0^2-3(0)-2=-2$ $(0,-2)$

Tangent Line: $y+2=-3(x-0)$ or $y+2=-3x$

Normal Line: $y+2=\frac{1}{3}(x-0)$ or $y+2=\frac{1}{3}x$

No need to simplify, but if you did.

Need slope at $x=1$

Need point $(1, f(1))$

10. Find an equation of the tangent line to the graph of $f(x)=\frac{3}{x}$ at $x=1$.

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{3}{1+h} - \frac{3(1+h)}{1(1+h)}}{h} = \lim_{h\to 0} \frac{3-3(1+h)}{1+h}\cdot\frac{1}{h} = \lim_{h\to 0} \frac{-3h}{1+h}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-3}{1+h} = \frac{-3}{1+0} = -3 \ldots \text{ slope of } f(x) \text{ at } x=1 \text{ is } -3.$$

point: $x=1$

$f(1)=\frac{3}{1}=3$ $\Big\}$ $(1,3)$

$\boxed{\text{Tangent Line: } y-3=-3(x-1)}$

11. An object is dropped from the top of a 150-m tower. It's height above the ground after t seconds is $150-4.9t^2$ m. How fast is the object falling 2 seconds after it is dropped?

$s(t)=150-4.9t^2$

This means: What is the instantaneous rate of chg. in position function when $t=2$ sec?

$$\lim_{h\to 0} \frac{s(2+h)-s(2)}{h} = \lim_{h\to 0} \frac{(150-4.9(2+h)^2)-(150-4.9(2)^2)}{h} = \lim_{h\to 0} \frac{(150-4.9(4+4h+h^2))-(150-19.6)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{150}-\cancel{19.6}-19.6h-4.9h^2-\cancel{150}+\cancel{19.6}}{h} = \lim_{h\to 0} \frac{-19.6h-4.9h^2}{h} = \lim_{h\to 0} \frac{h(-19.6-4.9h)}{h}$$

$$= \lim_{h\to 0} -19.6-4.9h$$

$$= -19.6-4.9(0)$$

$$= \boxed{-19.6 \text{ m/sec}}$$

12. What is the rate of change of the area of a circle with respect to the radius when the radius is 4 in?

IROC at $r=4$

$A=\pi r^2$

$$\lim_{h\to 0} \frac{A(4+h)-A(4)}{h} = \lim_{h\to 0} \frac{\pi(4+h)^2-\pi(4)^2}{h} = \lim_{h\to 0} \frac{\pi(16+8h+h^2)-16\pi}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{16\pi}+8\pi h+\pi h^2-\cancel{16\pi}}{h} = \lim_{h\to 0} \frac{8\pi h+\pi h^2}{h} = \lim_{h\to 0} \frac{h(8\pi+\pi h)}{h} = \lim_{h\to 0} 8\pi+\pi h$$

$$= 8\pi+\pi(0)$$

$$= \boxed{8\pi \ \frac{\text{in}^2}{\text{in.}} \text{ or } 8\pi \text{ in.}}$$

13. At what point is the tangent line to $g(x)=x^2-6x+1$ horizontal?

IROC

horizontal lines have slope 0

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2-6(x+h)+1)-(x^2-6x+1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{6x}-6h+\cancel{1}-\cancel{x^2}+\cancel{6x}-\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2-6h}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h-6)}{h}$$

$$= \lim_{h\to 0} 2x+h-6$$

$$= 2x+0-6$$

formula for slope of tan. line $= 2x-6$

point of tangency:

$x=3$

$g(3)=(3)^2-6(3)+1$

$g(3)=-8$

$\therefore$ The tangent line is horizontal at the point $(3,-8)$.

Slope $=0$

$2x-6=0$

$2x=6$

$x=3$

tangent line is horizontal when $x=3$