Worksheet 1. What You Need to Know About Motion Along the x-axis (Part 1)

In discussing motion, there are three closely related concepts that you need to keep straight. These are:

If $x(t)$ represents the position of a particle along the x-axis at any time t, then the following statements are true.

1. "Initially" means when ____________ = 0.
2. "At the origin" means ____________ = 0.
3. "At rest" means __________ = 0.
4. If the velocity of the particle is positive, then the particle is moving to the ____________.
5. If the velocity of the particle is __________, then the particle is moving to the left.
6. To find average velocity over a time interval, divide the change in _______ by the change in time.
7. Instantaneous velocity is the velocity at a single moment (instant!) in time.
8. If the acceleration of the particle is positive, then the __________ is increasing.
9. If the acceleration of the particle is __________, then the velocity is decreasing.
10. In order for a particle to change direction, the __________ must change signs.
11. One way to determine total distance traveled over a time interval is to find the sum of the absolute values of the differences in position between all resting points. Here's an example: If the position of a particle is given by:

$$x(t) = \frac{1}{3}t^3 - t^2 - 3t + 4,$$

find the total distance traveled on the interval $0 \le t \le 6$.

Worksheet 1. Solutions and Notes for Students

The three concepts are as follows.

Position: $x(t)$—determines where the particle is located on the x-axis at a given time t

Velocity: $v(t) = x'(t)$—determines how fast the position is changing at a time t as well as the direction of movement

Acceleration : $a(t) = v'(t) = x''(t)$—determines how fast the velocity is changing at time t; the sign indicates if the velocity is increasing or decreasing

The true statements are as follows.

1. "Initially" means when __**time, *t***__ = 0.
2. "At the origin" means __**position, *x*(*t*)**__ = 0.
3. "At rest" means __**velocity, *v*(*t*)**__ = 0.
4. If the velocity of the particle is positive, then the particle is moving to the __**right**__.
5. If the velocity of the particle is __**negative**__, then the particle is moving to the left.
6. To find average velocity over a time interval, divide the change in __**position**__ by the change in time.
7. Instantaneous velocity is the velocity at a single moment (instant!) in time.
8. If the acceleration of the particle is positive, then the __**velocity**__ is increasing.
9. If the acceleration of the particle is __**negative**__, then the velocity is decreasing.
10. In order for a particle to change direction, the __**velocity**__ must change signs.
11. First, find the times at which $x'(t) = v(t) = 0$. That would be $t = -1$ (which is out of our interval) and $t = 3$. Next, evaluate the position at the end points and at each of the "resting" points. The particle moved to the left 9 units and then to the right by 27 units for a total distance traveled of 36 units. Point out to students how this is similar to a closed interval test, where you have to determine function values at the end points as well as at any critical points found.

t	$x(t)$
0	4
3	−5
6	22

4 + 5 = 9
+
22 + −5 = 27
$\boxed{36}$

Worksheet 2. Sample Practice Problems for the Topic of Motion (Part 1)

Example 1 (numerical).

The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

2. Is there a time during the time interval $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer.

3. Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computations that lead to your answer and indicate units of measure.

4. Let $a(t)$ denote the acceleration of the particle at time t. Is there guaranteed to be a time $t = c$ in the interval $0 \le t \le 12$ such that $a(c) = 0$? Justify your answer.

Worksheet 2. Solutions

Example 1 (numerical)

1. At $t = 0$, the particle is moving to the left because the velocity is negative.

2. Yes, there is a time when the particle is at rest during the time interval $0 < t < 12$ minutes. Since the velocity function is differentiable, it also is continuous. Hence, by the Intermediate Value Theorem, since velocity goes from negative to positive, it must go through zero and $v(t) = 0$ means the particle is at rest.

3. Since $t = 10$ is not one of the times given in the table, we should approximate the derivative by using a difference quotient with the best (closest) data available. Because 10 lies between 8 and 12, the best approximation is given by:

$$v'(10) \approx \frac{v(12)-v(8)}{12-8} = \frac{5-7}{12-8} = -\frac{1}{2}\ \frac{\text{m/min}}{\text{min}} = -\frac{1}{2}\frac{\text{m}}{\text{min}^2}.$$

Here, $v'(10)$ is the acceleration of the particle at $t = 10$ minutes.

4. Yes, such a point is guaranteed by the Mean Value Theorem or Rolle's Theorem. Since velocity is differentiable (and therefore also continuous) over the interval $6 < t < 12$ and:

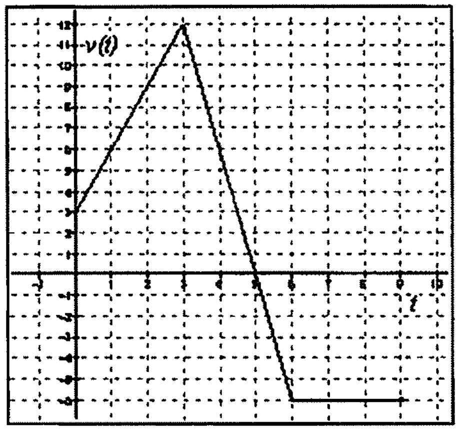
$$\frac{v(12)-v(6)}{12-6} = 0,$$

then there must exist a point c in the interval such that $v'(c) = a(c) = 0$.

Note: If we add the hypothesis that v' is continuous, then we may use the Intermediate Value Theorem to establish the result. In this case, since the values in the table indicate that velocity increases and then decreases on the interval $0 < t < 12$, then $v'(t) = a(t)$ must go from positive to negative and by the Intermediate Value Theorem must therefore pass through zero somewhere in that interval. It is the Mean Value Theorem, applied to the differentiable function v, that guarantees v' takes on at least one positive value in the interval $0 < t < 8$ (note that $\frac{7-(-3)}{8-0} = \frac{5}{4}$ is one such value), and at least one negative value in the interval $8 < t < 12$ (note that $\frac{5-7}{12-8} = -\frac{1}{2}$ is one such value).

Example 2 (graphical).

The graph below represents the velocity v, in feet per second, of a particle moving along the x-axis over the time interval from $t = 0$ to $t = 9$ seconds.



1. At $t = 4$ seconds, is the particle moving to the right or left? Explain your answer.

2. Over what time interval is the particle moving to the left? Explain your answer.

3. At $t = 4$ seconds, is the acceleration of the particle positive or negative? Explain your answer.

4. What is the average acceleration of the particle over the interval $2 \le t \le 4$? Show the computations that lead to your answer and indicate units of measure.

5. Is there guaranteed to be a time t in the interval $2 \le t \le 4$ such that $v'(t) = -3/2$ ft/sec^2? Justify your answer.

6. At what time t in the given interval is the particle farthest to the right? Explain your answer.

Example 3 (analytic).

A particle moves along the x-axis so that at time t its position is given by:

$$x(t) = t^3 - 6t^2 + 9t + 11$$

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

2. At $t = 1$, is the velocity of the particle increasing or decreasing? Explain your answer.

3. Find all values of t for which the particle is moving to the left.

4. Find the total distance traveled by the particle over the time interval $0 \le t \le 5$.

Example 2 (graphical)

1. At $t = 4$ seconds, the particle is moving to the right because the velocity is positive.

2. The particle is moving to the left over the interval $5 < t \le 9$ seconds because the velocity is negative.

3. The acceleration of the particle is negative because the velocity is decreasing, OR the acceleration is the slope of the velocity graph and the slope of the velocity graph at $t =$ 4 is negative.

4. Average acceleration over the time interval can be found by dividing the change in velocity by the change in time:

$$\frac{v(4)-v(2)}{4-2}=\frac{6-9}{4-2}=-\frac{3}{2}\frac{\text{ft/sec}}{\text{sec}}=-\frac{3}{2}\frac{\text{ft}}{\text{sec}^2}$$

5. No such time is guaranteed. The Mean Value Theorem does not apply since the function is not differentiable at $t = 3$ due to the sharp turn in the graph. If students have not yet learned the MVT, you can slide a tangent line (toothpick or stick) along the graph to show that no such point exists where the slope of the tangent line would be equal to the slope of a secant line between $t = 2$ and $t = 4$.

6. The particle is farthest to the right at $t = 5$ seconds. Since the velocity is positive during the time interval $0 \le t < 5$ seconds and negative during the time interval $5 < t \le 9$ seconds, the particle moves to the right before time $t = 5$ seconds and moves to the left after that time. Therefore, it is farthest to the right at $t = 5$ seconds.

Example 3 (analytic)

1. The particle is moving to the right because $x'(0) = v(0) = 9$ which is positive.

2. At $t = 1$, the velocity of the particle is decreasing because $x''(1) = v'(1) = a(1) = -6$, and if the acceleration is negative then the velocity is decreasing.

3. The particle is moving to the left for all values of t where $v(t) < 0$. We have:

$$v(t) = x'(t) = 3t^2 - 12t + 9 < 0 \text{ for } 1 < t < 3.$$

4.

t	$x(t)$
0	11
1	15
3	11
5	31

The particle moves right 4 spaces, left 4 spaces and then right 20 spaces. Therefore, the particle has traveled a total of 28 units. The common error that students make is to calculate $x(5)-x(0)=20$, which gives the displacement or net change in position, rather than the total distance traveled. Teachers should reinforce the difference between these two concepts every chance they get.

Worksheet 3. Understanding the Relationships Among Velocity, Speed, and Acceleration

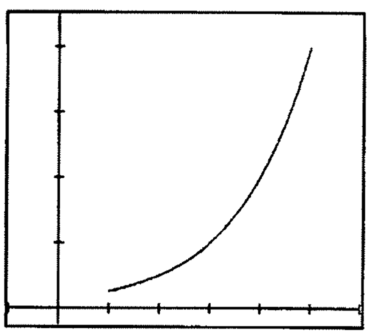
Speed is the absolute value of velocity. It tells you how fast something is moving without regard to the direction of movement.

1. What effect does absolute value have on numbers?

2. What effect does taking the absolute value of a function have on its graph?

For each situation below, the graph of a differentiable function giving velocity as a function of time t is shown for $1 \le t \le 5$, along with selected values of the velocity function. In the graph, each horizontal grid mark represents 1 unit of time and each vertical grid mark represents 4 units of velocity. For each situation, plot the speed graph on the same coordinate plane as the velocity graph and fill in the speed values in the table. Then, answer the questions below based on both the graph and the table of values.

Situation 1: Velocity graph



time	velocity	speed
1	1	
2	2	
3	4	
4	8	
5	16	

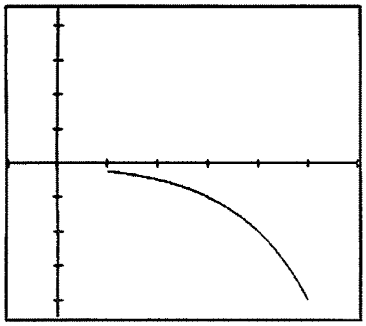
In this situation, the velocity is ____________ and ____________.
Positive or negative? Increasing or decreasing?

Because velocity is ____________, we know acceleration is ____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is

____________.
Increasing or decreasing?

Situation 2: Velocity graph



time	velocity	speed
1	-1	
2	-2	
3	-4	
4	-8	
5	-16	

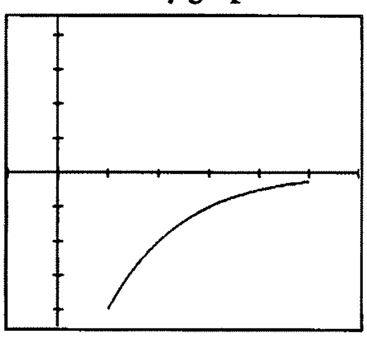
In this situation, the velocity is ____________ and ____________.
Positive or negative? Increasing or decreasing?

Because velocity is ______________, we know acceleration is ____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is

________________.
Increasing or decreasing?

Situation 3: Velocity graph



time	velocity	speed
1	-16	
2	-8	
3	-4	
4	-2	
5	-1	

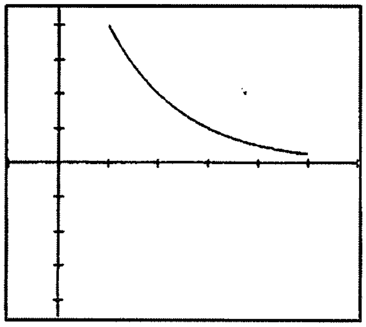
In this situation, the velocity is ____________ and ____________.
Positive or negative? Increasing or decreasing?

Because velocity is ______________, we know acceleration is ____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is

________________.
Increasing or decreasing?

Situation 4: Velocity graph



time	velocity	speed
1	16	
2	8	
3	4	
4	2	
5	1	

In this situation, the velocity is ____________ and ____________.
Positive or negative? Increasing or decreasing?

Because velocity is ____________, we know acceleration is ____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is ________________.
Increasing or decreasing?

Conclusion:

In which situations was the speed increasing? ______________________________

When the speed is increasing, the velocity and acceleration have ____________ signs.
Same or opposite?

In which situations was the speed decreasing? ______________________________

When the speed is decreasing, the velocity and acceleration have ____________ signs.
Same or opposite?

Assessing Students' Understanding (A Short Quiz):

1. If velocity is negative and acceleration is positive, then speed is ____________.

2. If velocity is positive and speed is decreasing, then acceleration is __________.

3. If velocity is positive and decreasing, then speed is _______________________.

4. If speed is increasing and acceleration is negative, then velocity is __________.

5. If velocity is negative and increasing, then speed is _______________________.

6. If the particle is moving to the left and speed is decreasing, then acceleration is
   ____________________.

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