Theme 6

SOLUTIONS

Multiple Choice

1. Answer (e). First use the Second Fundamental Theorem of Calculus to find $g'(x) = x^3e^x$. Differentiating this expression, $g''(x) = 3x^2e^x + x^3e^x$. Finally, $g''(1) = 3e + e = 4e$. You could have used integration by parts to explicitly find the antiderivative, but it would have made the problem much more difficult.

2. Answer (c).

$$h(x) = \int_{x^2}^{2} \sqrt{1+t^4}\,dt = -\int_{2}^{x^2} \sqrt{1+t^4}\,dt.$$

By the Second Fundamental Theorem of Calculus and the Chain Rule, $h'(x) = -\sqrt{1+(x^2)^4}\,(2x)$ and $h'(1) = -2\sqrt{2}$. Notice that the integrand does not have an elementary antiderivative.

3. Answer (e). $F(x)$ gives the area of the region under the semicircle $y = \sqrt{16 - t^2}$ between $t = -4$ and $t = x$. As x varies from -4 to 4, $F(x)$ goes from 0 to 8π, the area of a semicircle of radius 4. Verify this answer by graphing $F(x)$ on the viewing window $[-4, 5] \times [0, 30]$.

Free Response

(a) $F\left(-\frac{3}{2}\right) = \int_{-2}^{2(-3/2)+1} f(t)\,dt = \int_{-2}^{-2} f(t)\,dt = 0.$

(b) By the Second Fundamental Theorem of Calculus and the Chain Rule, $F'(x) = f(2x+1)(2)$ and $F'(0) = 2f(1) = 2$.

(c) Since the domain of f is $[-2, 2]$, solve the inequality $-2 \le 2x + 1 \le 2$ to obtain the domain of F, $[-3/2, 1/2]$.

(d) $F'(x) = 2f(2x+1) = 0$ when $2x + 1$ equals -2, 0, and 2, the zeros of f. So, the critical numbers of F are $-\frac{3}{2}$, $-\frac{1}{2}$, and $\frac{1}{2}$.

By checking the two intervals determined by these three points, you can see that F is decreasing on

$$\left(-\tfrac{3}{2}, -\tfrac{1}{2}\right)\ (F'(-1) = 2f(-1) = -2 < 0)$$

and increasing on

$$\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\ (F'(0) = 2 > 0).$$

So, the minimum occurs at $x = -\frac{1}{2}$.

SOLUTIONS

Multiple Choice

1. Answer (b). Set the derivative of the position function equal to zero: $v(t) = x'(t) = 1 + \ln 2t = 0$. This gives $\ln 2t = -1$ or $t = 1/(2e)$. The acceleration is $a(t) = v'(t) = 1/t$ and so $a(1/(2e)) = 2e$.
2. Answer (c). The total distance traveled by the particle is the integral of the speed. Because the graph of $v(t)$ is nonnegative on $[0, \pi/2]$, and negative on $(\pi/2, \pi)$ you must split up the integral as follows.

$$\text{total distance} = \int_0^{\pi} |\sin 2t|\, dt = \int_0^{\pi/2} \sin 2t\, dt + \int_{\pi/2}^{\pi} (-\sin 2t)\, dt = 1 + 1 = 2.$$

You can verify this answer by integrating $y = |\sin 2t|$ from $t = 0$ to $t = \pi$ with a graphing utility. Notice that the particle has returned to its original position.

3. Answer (e). You can obtain the velocity by integrating the acceleration function using integration by parts.

$$v(t) = \int a(t)\, dt = \int te^{2t}\, dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$$

Since $v(0) = -1/4$, $C = 0$. The speed at $t = 1/4$ is the absolute value of the velocity.

$$\text{speed} = \left|v\left(\frac{1}{4}\right)\right| = \left|\frac{1}{8}e^{1/2} - \frac{1}{4}e^{1/2}\right| = \left|-\frac{\sqrt{e}}{8}\right| = \frac{\sqrt{e}}{8}$$

Free Response

(a) $v(t) = \int a(t)\, dt = \int \pi \cos \pi t\, dt = \sin \pi t + C$. Since $v(1/2) = 1/2$, $1/2 = \sin(\pi/2) + C$, which gives $C = -1/2$. So, $v(t) = \sin \pi t - 1/2$.

(b) Since $\sin \pi t \geq -1$, the minimum velocity is $-3/2$.

(c) Integrating the velocity function, you have

$$x(t) = \int v(t)\, dt = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + C_1.$$

Because $x(0) = 0$, $C_1 = 1/\pi$ and

$$x(t) = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi}.$$

(d) The particle returns to the origin when

$$\frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi} = 0.$$

By graphing the function

$$y = \frac{-\cos \pi t}{\pi} - \frac{t}{2} + \frac{1}{\pi}$$

on the interval $[0, 2]$, you see that the first positive zero of y is approximately $t = 0.353$.