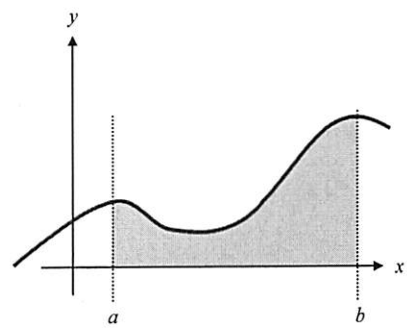
The Area Problem and the Rectangular Approximation Method (RAM)
(a.k.a. Riemann Sums)



Suppose we wanted to know the area of the region bounded by a curve, the x – axis, and the lines $x = a$ and $x = b$, as shown at the right.

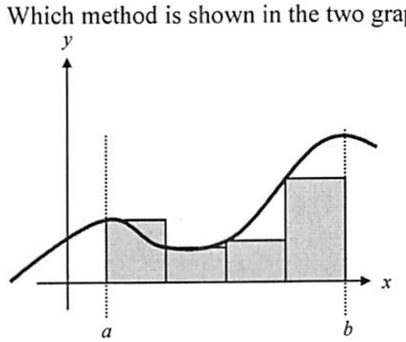
The first step is to divide the interval from a to b into subintervals. (The examples below show 4 and 8 subintervals, respectively.)

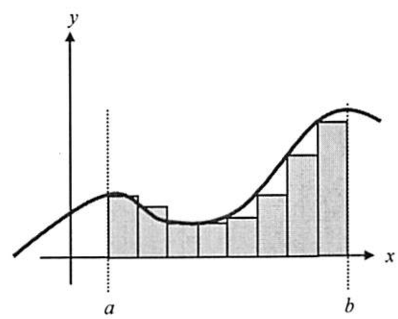
After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval as the base.

The *height of each rectangle is determined by the function value at a point in the specific subinterval*, and can be determined using 3 different methods.

We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM).

Example 4: Which method is shown in the two graphs below?





Example 5: The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

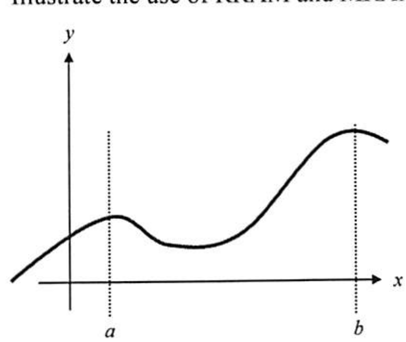
Summary of the Process: A sketch is almost mandatory!

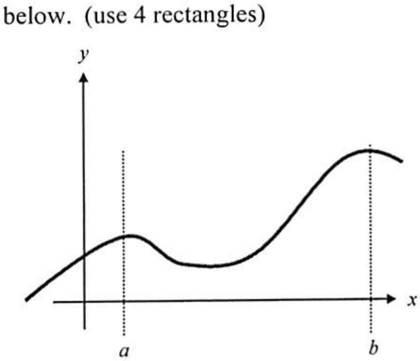
Step 1: Divide (or Partition) the interval into n subintervals.

Step 2: Create n rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

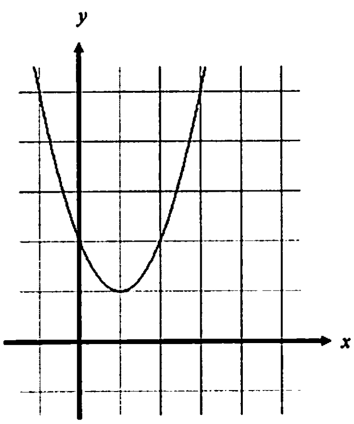
Step 3: Find the area of all n rectangles and add them together.

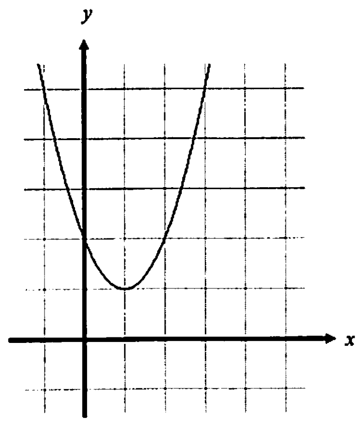
Example 6: Illustrate the use of RRAM and MRAM on the graphs below. (use 4 rectangles)

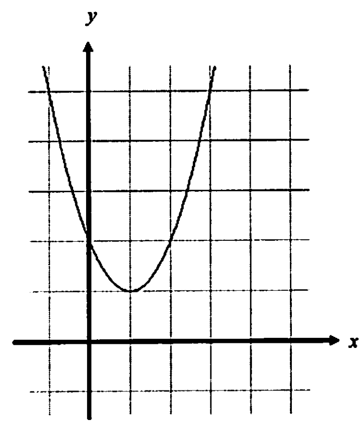




Example 7: Use 4 rectangles to approximate the area under the graph of $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$. Use LRAM, RRAM, and then MRAM.







Example 8: Using your rectangles as a guide, find each approximation.

a) LRAM =

b) RRAM =

c) MRAM =

Example 9: It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds.

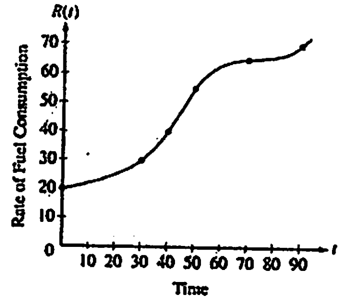
Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

a) Using a left Riemann Sum with 10 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

b) Using a Midpoint Riemann Sum with 5 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

2003
Q 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

2006
Q #4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

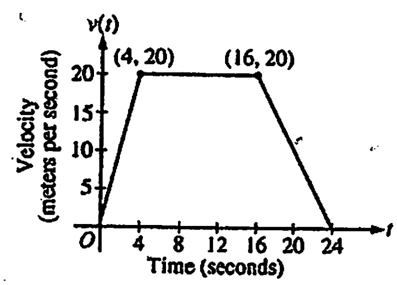
Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

2005
Q#5

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.



(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

LESSON 3

The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function R of time t. The table below shows the rate at selected values of t for a 12-hour period.

t (hrs)	0	2	4	6	8	10	12
$R(t)$ (gal/hr)	12.5	13.4	13.9	14.3	14.6	14.8	14.7

1. Use a midpoint Riemann sum with three subintervals to approximate:

$$\int_0^{12} R(t)\,dt.$$

Particle A moves along a horizontal line with a velocity $v_A(t)$, where $v_A(t)$ is a positive continuous function of t. The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity $v_A(t)$ of the particle at selected times is given in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (cm/sec)	1.7	6.8	7.4	15.6	24.9

1. Use data from the table to approximate the distance traveled by particle A over the interval $0 \le t \le 10$ seconds by using a right Riemann sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

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x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

5. Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.
   (a) Estimate $f'(4)$. Show the work that leads to your answer.
   (b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.
   (c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.
   (d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

2007

2008 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2. Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

   (a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

   (b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

   (c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

   (d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

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t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$.

   (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

   (a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

   (b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

   (c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

   (d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

Lesson 4

Watch Mr Leckie from the 9:22 minute marker to the end
Follow along with attached worksheets- then do page 3
http://www.chaoticgolf.com/tutorials_calc_ch5.htmlesson 4

5.2 Definite Integrals

Notation for Definite Integrals

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle, Δx, goes to zero. As we did in the section on differentials, we are going to use the notation *dx* to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an *Integral Sign*, $\int$, which looks somewhat like a giant "S" for sum.

The $f(c_k)$ which represented a different function value for each interval is going to be replaced with $f(x)$ since the x – values are going to be soooooooo close together it's almost as if we are evaluating the function at EVERY x – value in the interval $[a, b]$. Combining all of this we have the following notation:

$$\int_a^b f(x)\,dx$$

We read the notation above as "***<u>The Integral of f of x from a to b</u>***"

Important ♪ (Actually it's a Theorem): IF the function is continuous, THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

Using Definite Integrals as Area

We can define the **area under the curve** $y = f(x)$ **from** a **to** b as an *integral* from a to b ...
... AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval $[a, b]$.

Drawing a picture and using geometry is still a valid method of finding areas in this class!

Example 1: For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

a) $\int_2^9 3\,dx$ b) $\int_{-2}^1 |x|\,dx$. c) $\int_{-3}^3 \sqrt{9-x^2}\,dx$

1

So ... what happens if the "area" is below the x-axis ... as I mentioned before, "area" is inherently positive, but a Riemann sum ... and therefore an Integral can have negative values if the curve lies below the x-axis.

Example 2: Consider the function $f(x) = 3 - x$. Sketch a graph of this function.

a) What is the "AREA" between the curve and the x – axis between $x = 4$ and $x = 8$?

b) Evaluate $\int_4^8 (3-x)\,dx$

Example 3: Given $\int_0^\pi \sin x\,dx = 2$, use what you know about a sine function to evaluate the following integrals.

a) $\int_\pi^{2\pi} \sin x\,dx$

b) $\int_0^{2\pi} \sin x\,dx$

c) $\int_0^{\pi/2} \sin x\,dx$

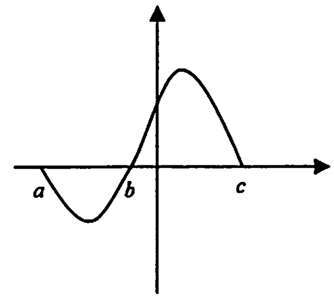
d) $\int_{-\pi}^{\pi} \sin x\,dx$

e) $\int_0^\pi (2+\sin x)\,dx$

(2)

1. Graphically speaking, if $f(x)$ is always above the x-axis, what does $\int_a^b f(x)\,dx$ mean?

2. Given the graph of $f(x)$ below, answer the following questions:



a) Is $\int_a^b f(x)\,dx$ positive, negative, or zero? Why?

b) Is $\int_b^c f(x)\,dx$ positive, negative, or zero? Why?

c) Is $\int_a^c f(x)\,dx$ positive, negative, or zero? Why?

3. Use your knowledge of the graph of $y = x^3$, your understanding of area, and the fact that $\int_0^1 x^3\,dx = \frac{1}{4}$ to answer the following: (Draw a sketch for each one!)

a) $\int_{-1}^{1} x^3\,dx$

b) $\int_0^1 \left(x^3+3\right)dx$

c) $\int_0^1 \left(x^3-1\right)dx$

4. Draw a sketch and shade the "area" indicated by each integral, then use geometry to evaluate each integral.

a) $\int_{1/2}^{3/2} (-2x+4)\,dx$

b) $\int_{-4}^{0} \sqrt{16-x^2}\,dx$

c) $\int_{-1}^{1} (2-|x|)\,dx$

5. If $\int_2^5 f(x)\,dx = 18$, then $\int_2^5 (f(x)+4)\,dx = ?$

6. Use areas to evaluate $\int_a^b 2s\,ds$, where a and b are constants and $0 < a < b$

7. Which of the following quantities would NOT be represented by the definite integral $\int_0^8 70\,dt$?

A) The distance traveled by a train moving 70 mph for 8 minutes
B) The volume of ice cream produced by a machine making 70 gallons per hour for 8 hours
C) The length of a track left by a snail traveling at 70 cm per hour for 8 hours
D) The total sales of a company selling $70 of merchandise per hour for 8 hours
E) The amount the tide has risen 8 min after low tide if it rises at a rate of 70 mm per minute during that period

8. Express the desired quantity as a definite integral and then evaluate using geometry.

a) Find the distance traveled by a train moving at 87 mph from 8:00 AM to 11:00 AM

b) Find the output from a pump producing 25 gallons per minute during the first hour of its operation.

c) Find the calories burned by a walker burning 300 calories per hour between 6:00 PM and 7:30 PM.

d) Find the amount of water lost from a bucket leaking 0.4 liters per hour between 8:30 AM and 11:00 AM.

9. Draw a sketch for the area enclosed between the x-axis and the graph of $y = 4 - x^2$ from $x = -2$ to $x = 2$.

a) Set up a definite integral to find the area of the region.

b) Use your calculator to evaluate the integral expression you set up in part a.