4 The Graphical Relationship Between First and Second Derivatives

Key to Text Coverage

Section	Examples	Exercises	Topics
2.1	1–7	39–45, 49, 50	*Relationship between the graphs of f and f'*
3.3	1–4	43–48	*Increasing and decreasing functions*
3.4	1–4	45–56	*Concavity and second derivatives*
3.6	1–6	1–6, 51–58	*General curve sketching principles*

Formulas

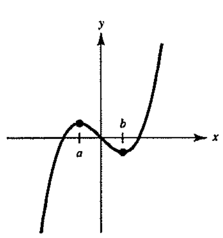
Let $y = f(x)$ be defined on an open interval I containing c.

- f is increasing on I if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- f is decreasing on I if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
- If $f'(x) > 0$ then f is increasing on I.
- If $f'(x) < 0$ then f is decreasing on I.
- If f' changes from negative to positive at c, there is a relative minimum at c.
- If f' changes from positive to negative at c, there is a relative maximum at c.
- f is concave upwards on I if f' is increasing on I.
- f is concave downwards on I if f' is decreasing on I.
- If $f'' > 0$ on I then f is concave upwards on I.
- If $f'' < 0$ on I then f is concave downwards on I.
- The point $(c, f(c))$ is a point of inflection if the concavity changes there.

Summary

The first and second derivatives of a function provide an enormous amount of useful information about the shape of the graph of the function, as indicated by the properties above. An important skill to develop is that of producing the graph of the derivative of a function, given the graph of the function. Conversely, it is important to be able to produce the graph of a function given the graph of its derivative.

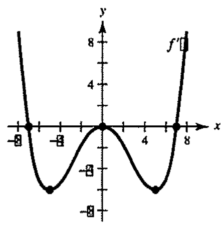
For instance, consider the graph of f below. Because $f'(x) > 0$ on the interval $(-\infty, a)$, f is increasing on that interval. Furthermore $f'(x)$ is decreasing near $x = a$, which implies that the graph of f is concave downwards near $x = a$.



Theme 4

Worked Example

Consider the graph of f', the derivative of $y = f(x)$ defined on the domain $-9 < x < 9$.



(a) For what values of x does f have a relative minimum?

(b) For what values of x does f have a relative maximum?

(c) Determine the open intervals where the graph of f is concave downwards. Show the analysis that leads to your conclusion.

(d) Sketch the graph of f on the interval $(-9, 9)$ if $f(0) = 0$. Show the analysis that leads to your graph.

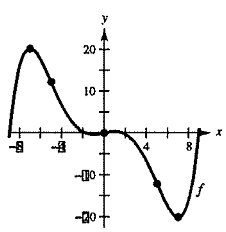
SOLUTION

(a) There is a relative minimum at $x = 7$.

(b) There is a relative maximum at $x = -7$.

(c) The graph of f' is decreasing on the intervals $(-9, -5)$ and $(0, 5)$. By the definition of concavity, this means that the graph of f is concave downwards on these intervals.

(d) The graph of f is increasing to the left of $x = -7$. The graph is decreasing on the intervals $(-7, 0)$ and $(0, 7)$, and increasing on the interval $(7, 9)$. There is an inflection point at $x = 0$ because the concavity changes from concave upwards to concave downwards.



Notes

(a) The critical numbers are those values of x for which $f'(x) = 0$ or is undefined. From the graph of f' you see that the critical numbers are $x = -7, 0, 7$. Because f' changes from negative to positive at $x = 7$, there is a relative minimum at $x = 7$.

(b) f' changes from positive to negative at $x = -7$, which means that there is a relative maximum at $x = -7$.

(c) In a similar manner, f is concave upwards on the intervals $(-5, 0)$, $(5, 9)$. Since the concavity changes at the points $x = -5, 0, 5$, these are the x-coordinates of points of inflection.

Name ______________________________

Date ______________________________

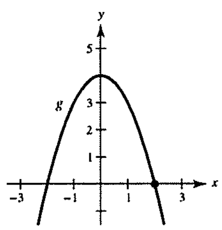
Sample Questions

Show all your work on a separate sheet of paper. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your answers.

Multiple Choice

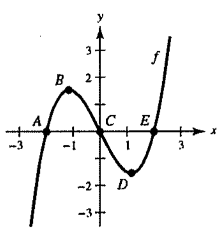
1. Given the graph of $y = g(x)$, estimate the value of $g'(2)$.

(a) -4 (b) -1 (c) 0 (d) 1 (e) 4



2. At which point *A, B, C, D,* or *E* on the graph of $y = f(x)$ are both y' and y'' positive?

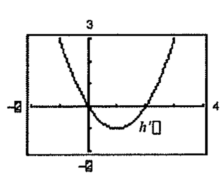
(a) *A* (b) *B* (c) *C* (d) *D* (e) *E*



3. Given the graph of $h'(x)$, which of the following statements are true about the graph of h?

I. The graph of h has a point of inflection at $x = 1$.
II. The graph of h has a relative extremum at $x = 0$.
III. The graph of h has a relative extremum at $x = 1$.

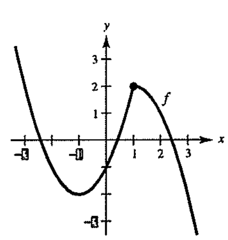
(a) I only (b) II only (c) III only (d) I and II only (e) I and III only



Free Response

The graph of the function f is shown in the figure.

(a) Estimate $f'(0)$.
(b) On what open intervals is f increasing?
(c) On what open intervals is f concave downwards?
(d) What are the critical numbers of f?
(e) Sketch the graph of f'.



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For each question, sketch a possible graph for f(x) based on the given information. Label all zeros, critical points and inflection points.
[Hint: Write out the sign analysis for f ' and f " before drawing the graph.]

1. f(x) is continuous and differentiable at all points
   $f(-3) = f(5) = 0$
   $f' > 0$ for $x < -1$ and for $x > 4$
   $f' < 0$ for $-1 < x < 4$
   $f'' < 0$ for $x < 0$
   $f'' > 0$ for $x > 0$

2. f(x) is continuous at all points
   $f(1) = 0$
   $f' > 0$ for all x except at $x = 1$
   f' is undefined at $x = 1$
   $f'' > 0$ for $x < 1$
   $f'' < 0$ for $x > 1$

3. f(x) is a piece-wise function that is continuous at all points
   $f(-1) = f(5) = 0$
   $f' > 0$ for all $x < 3$ except at $x = -1$
   $f' < 0$ for $x > 3$
   f' is undefined at $x = -1$
   $f' = 0$ at $x = 3$
   $f'' = 0$ for $x < -1$
   $f'' < 0$ for $x > -1$

Exercises and Problems for Section 2.6

Exercises

1. For the function graphed in Figure 2.52, are the following quantities positive or negative?

   (a) $f(2)$ (b) $f'(2)$ (c) $f''(2)$

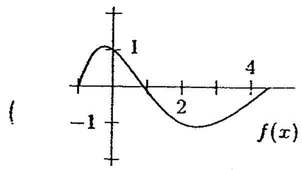
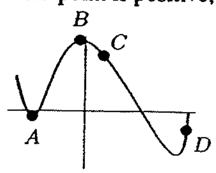


Figure 2.52

2. The graph of a function $f(x)$ is shown in Figure 2.53. On a copy of the table indicate whether f, f', f'' at each marked point is positive, negative, or zero.



Point	f	f'	f''
A			
B			
C			
D			

Figure 2.53

3. At which of the labeled points on the graph in Figure 2.54 are both dy/dx and d^2y/dx^2 positive?

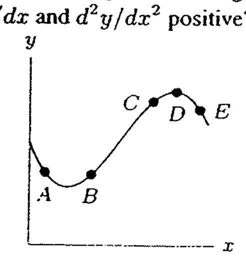


Figure 2.54

4. The distance of a car from its initial position t minutes after setting out is given by $s(t) = 5t^2 + 3$ kilometers. What are the car's velocity and acceleration at time t? Give units.

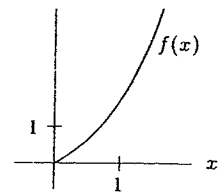
5. Sketch the graph of a function whose first and second derivatives are everywhere positive.

6. Sketch the graph of a function whose first derivative is everywhere negative and whose second derivative is positive for some x-values and negative for other x-values.

7. Sketch the graph of the height of a particle against time if velocity is positive and acceleration is negative.

For Exercises 8–13, give the signs of the first and second derivatives for each of the following functions.

8.



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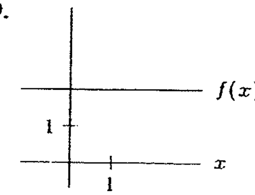

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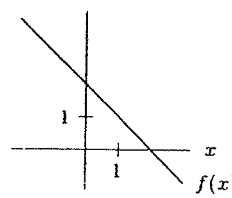

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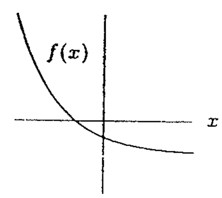

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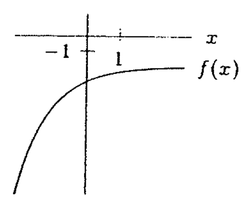

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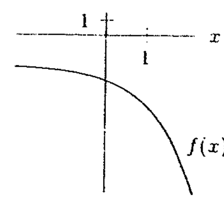

Problems

14. The table gives the number of passenger cars, $C = f(t)$, in millions, in the US in the year t.

    (a) Do $f'(t)$ and $f''(t)$ appear to be positive or negative during the period 1940–1980?

    (b) Estimate $f'(1975)$. Using units, interpret your answer in terms of passenger cars.

t (year)	1940	1950	1960	1970	1980
C (cars, in millions)	27.5	40.3	61.7	89.3	121.6

15. An accelerating sports car goes from 0 mph to 60 mph in five seconds. Its velocity is given in the following table, converted from miles per hour to feet per second, so that all time measurements are in seconds. (Note: 1 mph is 22/15 ft/sec.) Find the average acceleration of the car over each of the first two seconds.

Time, t (sec)	0	1	2	3	4	5
Velocity, $v(t)$ (ft/sec)	0	30	52	68	80	88

· DRAWING GRAPHS OF F(X)

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x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	−1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	−3	Negative
$f''(x)$	−2	Negative	0	Positive	DNE	Negative	0	Positive

4. Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.

GRAPH OF DERIVATIVE

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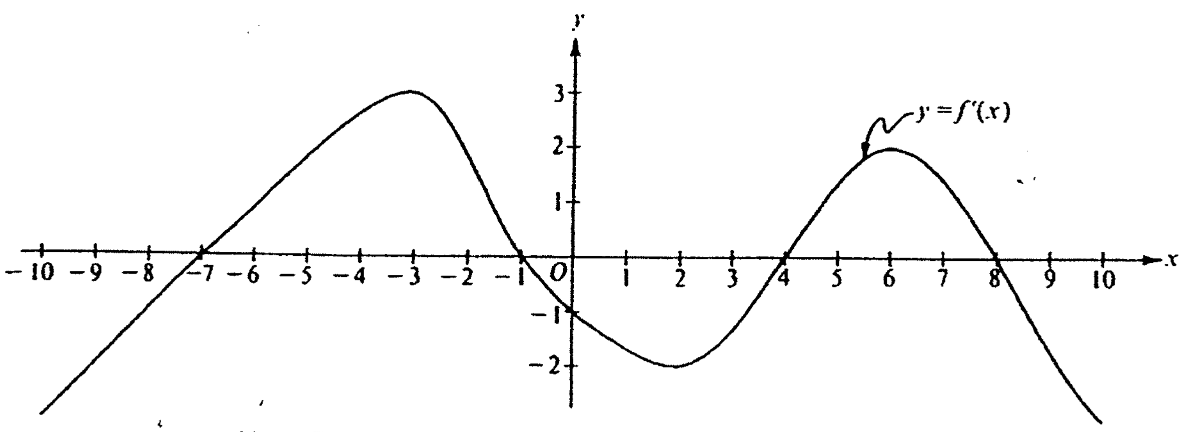

Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leq x \leq 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10, 10)$ does f have a relative maximum? Justify your answer.

(c) For what values of x is the graph of f concave downward?

(d) Draw the graph of f and f''