AP Calculus
3.1 Worksheet

Cofield

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What is the original limit definition of a derivative?

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

2. What is the alternative definition of a derivative?

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

3. Use the original definition of the derivative to find the derivative of each function at the indicated point.

a) $f(x)=\frac{1}{x}$ at $a=2$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x}-\cancel{x}-h}{x(x+h)}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{-1}{x(x+h)}=\frac{-1}{x(x+0)}$$

So $f'(x)=\frac{-1}{x^2}$ at any pt $x=a$. $\boxed{f'(2)=\frac{-1}{4}}$

b) $k(x)=3-x^2$ at $a=-2$.

$$k'(x)=\lim_{h\to 0}\frac{[3-(x+h)^2]-[3-(x)^2]}{h}$$

$$=\lim_{h\to 0}\frac{[3-(x^2+2xh+h^2)]-[3-x^2]}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3}-\cancel{x^2}-2xh-h^2-\cancel{3}+\cancel{x^2}}{h}=\lim_{h\to 0}\frac{h(-2x-h)}{h}$$

$$=\lim_{h\to 0}(-2x-h)=-2x-0$$

So $k'(x)=-2x$ at any $x=a$. $\boxed{k'(-2)=4}$

Remember you can also apply the point $x=a$ from the beginning!

c) $g(x)=\sqrt{x+1}$ at $a=3$

$$g'(3)=\lim_{h\to 0}\frac{g(3+h)-g(3)}{h}=\lim_{h\to 0}\frac{\sqrt{(3+h)+1}-\sqrt{3+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}\cdot\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2}=\lim_{h\to 0}\frac{h+\cancel{4}-\cancel{4}}{h(\sqrt{h+4}+2)}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{h+4}+2)}=\lim_{h\to 0}\frac{1}{\sqrt{h+4}+2}=\frac{1}{4}$$

$\boxed{\therefore\ g'(3)=\frac{1}{4}}$

d) $f(x)=x-x^3$ at $a=-1$

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{[(-1+h)-(-1+h)^3]-[-1-(-1)^3]}{h}$$

$$=\lim_{h\to 0}\frac{-1+h-(-1+3h-3h^2+h^3)-\cancel{(-1+1)}^{\,0}}{h}$$

$$=\lim_{h\to 0}\frac{-1+h+1-3h+3h^2-h^3}{h}=\lim_{h\to 0}\frac{-2h+3h^2-h^3}{h}$$

$$=\lim_{h\to 0}\frac{h(-2+3h-h^2)}{h}=\lim_{h\to 0}(-2+3h-h^2)=-2+3(0)-(0)^2$$

$$=-2$$

$\boxed{\therefore\ f'(-1)=-2}$

4. Repeat question 3a – 3c using the alternative definition of the derivative.

a) $$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$$

$$=\lim_{x\to 2}\frac{2-x}{2x}\cdot\frac{1}{x-2}=\lim_{x\to 2}\frac{-x+2}{2x}\cdot\frac{1}{x-2}$$

$$=\lim_{x\to 2}\frac{-1(x-2)}{2x}\cdot\frac{1}{x-2}=\lim_{x\to 2}\frac{-1}{2x}=\frac{-1}{2(2)}$$

$\boxed{\therefore\ f'(2)=\frac{-1}{4}}$

b) $$k'(-2)=\lim_{x\to -2}\frac{k(x)-k(-2)}{x-(-2)}=\lim_{x\to -2}\frac{(3-x^2)-(3-(-2)^2)}{x-(-2)}$$

$$=\lim_{x\to -2}\frac{3-x^2-(3-4)}{x+2}=\lim_{x\to -2}\frac{3-x^2+1}{x+2}$$

$$=\lim_{x\to -2}\frac{-(x^2-4)}{x+2}=\lim_{x\to -2}\frac{-\cancel{(x+2)}(x-2)}{\cancel{x+2}}$$

$$=\lim_{x\to -2}-(x-2)=\lim_{x\to -2}-x+2=-(-2)+2$$

$$=4$$

$\boxed{k'(-2)=4}$

Again we can find $g'(a)$ first, then evaluate at $a=3$.

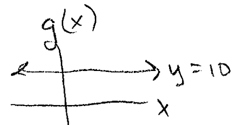
c) $$g'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(\sqrt{x+1}-\sqrt{a+1})}{(x-a)}\cdot\frac{(\sqrt{x+1}+\sqrt{a+1})}{(\sqrt{x+1}+\sqrt{a+1})}$$

$$=\lim_{x\to a}\frac{(x+1)-(a+1)}{(x-a)(\sqrt{x+1}+\sqrt{a+1})}=\lim_{x\to a}\frac{x+1-a-1}{(x-a)(\sqrt{x+1}+\sqrt{a+1})}$$

$$=\lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x+1}+\sqrt{a+1})}=\lim_{x\to a}\frac{1}{\sqrt{x+1}+\sqrt{a+1}}=\frac{1}{\sqrt{a+1}+\sqrt{a+1}}=\frac{1}{2\sqrt{a+1}}$$

So "derivative of g(x) w/ respect to x" evaluated at $x=3$ or $\left.\frac{d}{dx}(\sqrt{x+1})\right|_{x=3}=\frac{1}{2\sqrt{3+1}}=\boxed{\frac{1}{4}}$

5. Consider the function $g(x)=10$.



a) Using what you know about the graph of $g(x)$, what does $g'(6)=?$

$g'(6)=0$ b/c at $x=6$, slope $=0$.

b) Use the alternative definition AND the original definition of the derivative to verify your answer for $g'(6)$. YES … use BOTH definitions ☺

{original}

$$g'(6)=\lim_{h\to0}\frac{g(6+h)-g(6)}{h}$$
$$=\lim_{h\to0}\frac{10-10}{h}$$
$$=\lim_{h\to0}\frac{0}{h}$$
$$=\lim_{h\to0}0$$
$$=0\ \checkmark$$

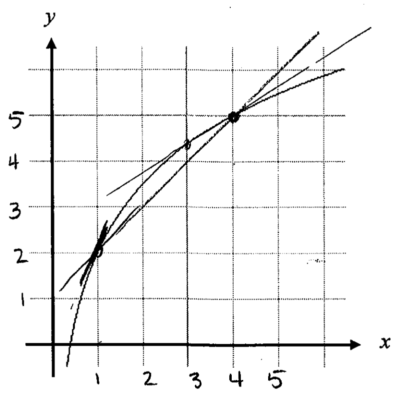
{alternate}

$$g'(6)=\lim_{x\to6}\frac{g(x)-g(6)}{x-6}$$
$$=\lim_{x\to6}\frac{10-10}{x-6}$$
$$=\lim_{x\to6}\frac{0}{x-6}$$
$$=\lim_{x\to6}0$$
$$=0\ \checkmark$$

6. If $\underbrace{f(2)=3}_{\text{point } (2,3)}$ and $\underbrace{f'(2)=5}_{\text{slope of } f(x) \text{ when } x=2 \text{ is } 5}$, find an equation of the tangent line at the point where $x=2$.

$$y-3=5(x-2)$$

7. Use the figure to the right to answer the following questions:



a) What is $f(1)$ and $f(4)$?

$f(1)=2$

$f(4)=5$

b) What is the geometric interpretation of $\frac{f(4)-f(1)}{4-1}$?

$\frac{f(4)-f(1)}{4-1}$ is the slope of the secant line between $(1,2)$ & $(4,5)$

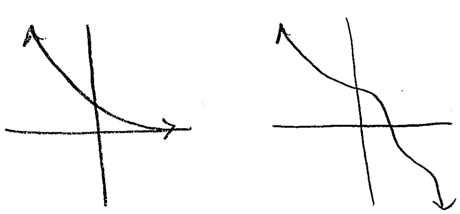
c) Using the geometric interpretation of each expression, insert the proper inequality symbol (< or >).

i) $\underbrace{\frac{f(4)-f(1)}{4-1}}_{\text{steeper positive slope}}\ \boxed{>}\ \frac{f(4)-f(3)}{4-3}$

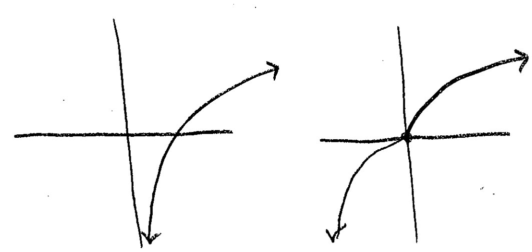
ii) $\underbrace{\frac{f(4)-f(1)}{4-1}}_{\text{slope of secant between } x=1,\,x=4}\ \boxed{}\ \underbrace{f'(1)}_{\text{slope of tangent at } x=1}$

8. Sketch a function whose derivative is ALWAYS negative and another whose derivative is ALWAYS positive.

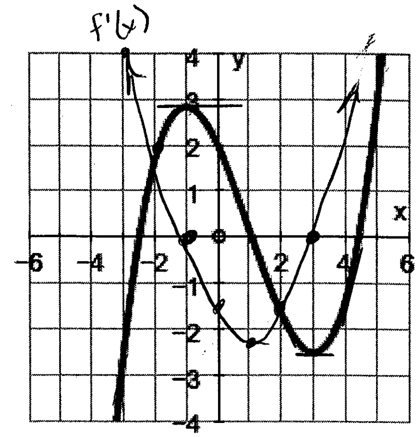
f' is always negative b/c function always decreasing.



f' is always positive b/c f always increasing



9. Use the graph of $f(x)$ shown to the right.



a) Where is $f'(x)=0$? Explain.

$f'(x)=0$ when $x=-1$ and $x=3$
b/c at those values the tangent is horizontal.

b) Where is $f'(x)>0$? Explain.

$f'(x)>0$ when x is on interval $(-\infty,-1)\cup(3,\infty)$
because that is where $f(x)$ is increasing.

c) Where is $f'(x)<0$? Explain.

$f'(x)<0$ when x is on interval $(-1,3)$ because
that is where $f(x)$ is decreasing.

d) On the same graph, draw a possible sketch of $f'(x)$.

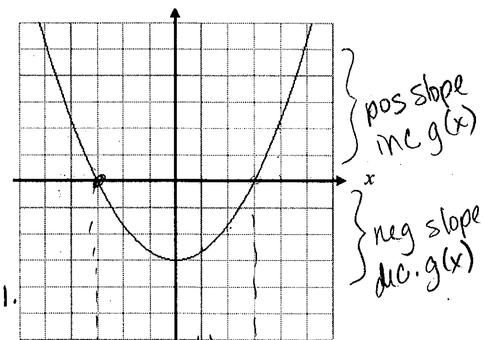
① plot $f'(x)=0$ for any x values where tangent is horizontal (there are 2)

② For any x values where $f(x)$ is increasing, $f'(x)$ is above x-axis.
For any x values where $f(x)$ is decreasing, $f'(x)$ is below x-axis.

③ approximate slopes
→ large pos
→ small pos
→ large neg
→ small neg

10. The figure to the right shows the graph of $g'(x)$.

The graph of $g'(x)$

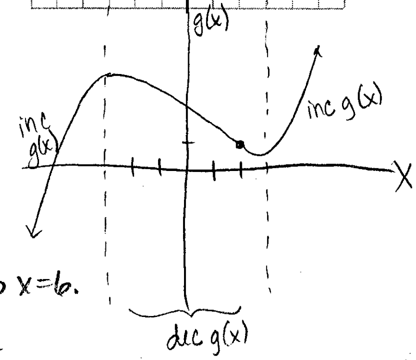


a) What does $g'(0)=$? Read graph How about $g'(3)$?

$g'(0)=-3$ $\qquad$ $g'(3)=0$

c) From the graph, $g'(1)=-\frac{8}{3}$. What does this tell us about the graph of g?

Since $g'(1)$ is negative, we know $g(x)$ is decreasing @ $x=1$.



d) From the graph, $g'(4)=\frac{7}{3}$. What does this tell us about the graph of g?

Since $g'(4)$ is positive, we know $g(x)$ is
increasing at $x=4$

e) Is $g(6)-g(4)$ positive or negative (those are g values not g')? Explain.

On the graph of $g'(x)$... $g'(x)>0$ for all values from $x=4$ to $x=6$.
Since the slopes are positive, then $g(x)$ is increasing from
$x=4$ to $x=6$. $\therefore$ $g(6)>g(4)$ b/c $g(x)$ is increasing and $g(6)-g(4)$ must be positive.

f) Find (if they exist) any value(s) of x, where $g'(x)=0$?

$g'(x)=0$ when $x=-3$ and $x=3$ ... read given graph!

g) Is it possible to find $g(2)$ from this graph? Explain.

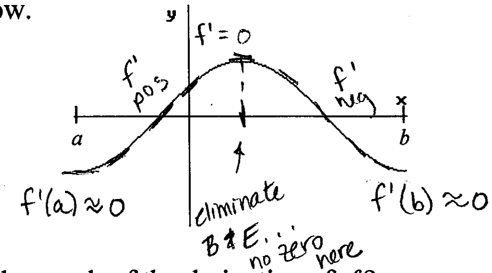
It is not possible to find $g(2)$ from the graph of $g'(x)$ because
$g'(x)$ only shows the slope values... not the output values.

h) What interval is $g(x)$ increasing? What interval is $g(x)$ decreasing? How do you know?

$g(x)$ increases on $(-\infty,-3]\cup[3,\infty)$ because the graph of $g'(x)$ is positive.
$g(x)$ decreases on $[-3,3]$ because the graph of $g'(x)$ is negative.

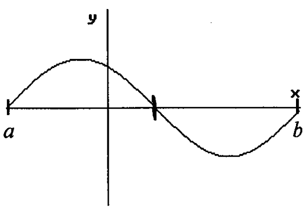
i) If you were told that $g(2)=1$, sketch a possible graph of $g(x)$? See above

11. The graph of f is shown below.

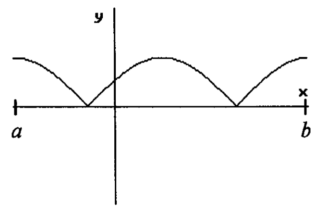


Which of the following could be the graph of the derivative of f?

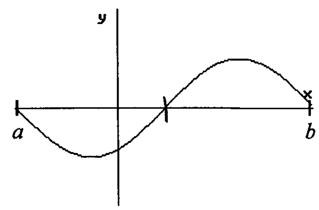
A.



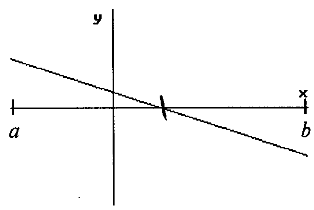
B.



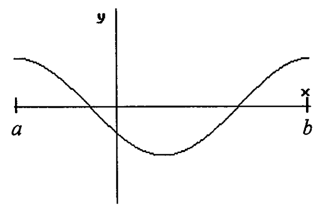
C.



D.

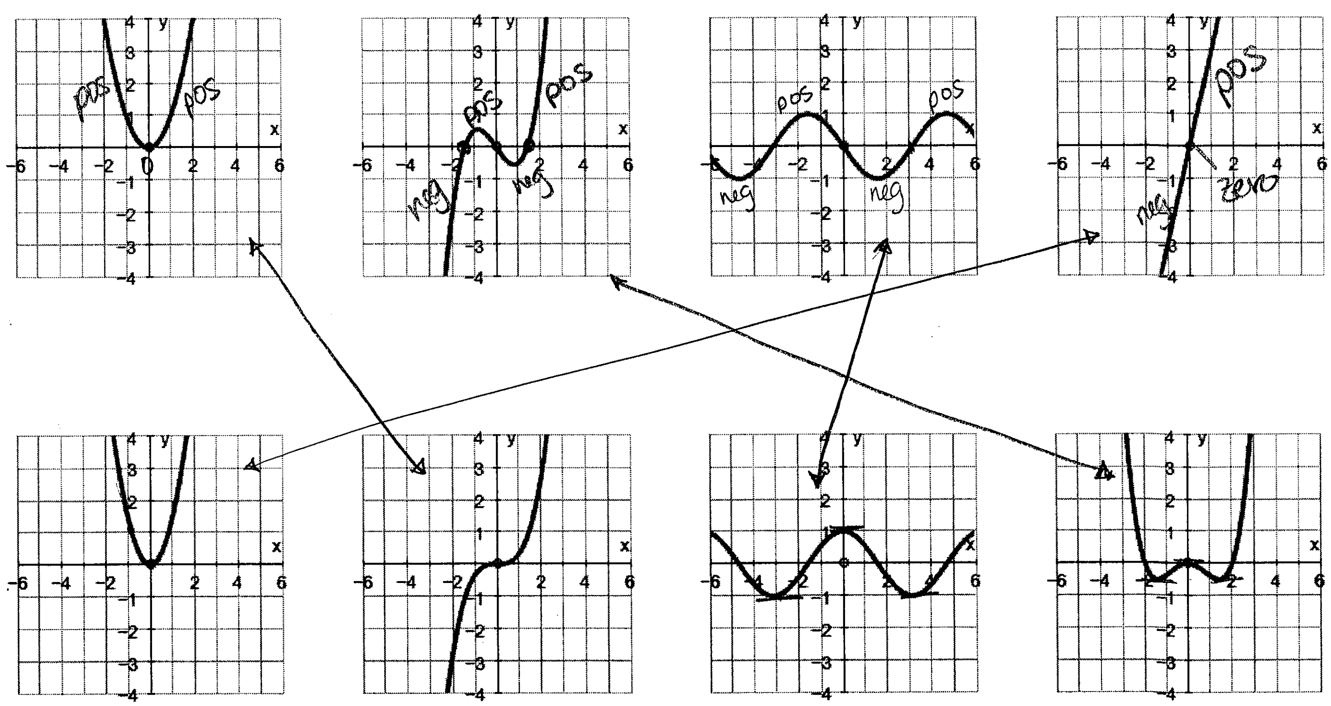


E.



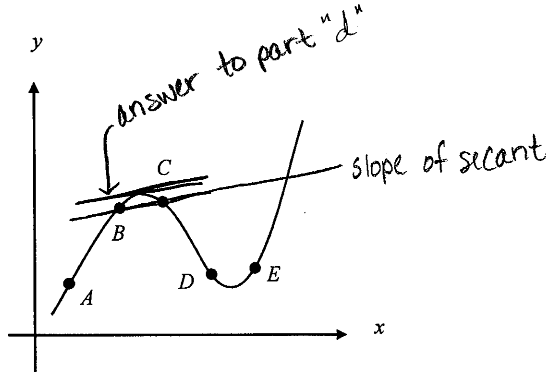
12. The graphs in the first row are the derivatives. Match them with the graph of their function shown in the second row.

(Graphs of Derivative)



(Graphs of Function)

13. Use the graph of f below to answer each question.

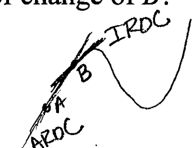


a) Between which two consecutive points is the average rate of change of the function greatest?

Avg. rate of change greatest from A to B because $\overline{AB}$ has the steepest slope.

b) Is the average rate of change between A and B greater than or less than the instantaneous rate of change of B?

avg. rate of change between A & B $>$ instantaneous rate of change at B.



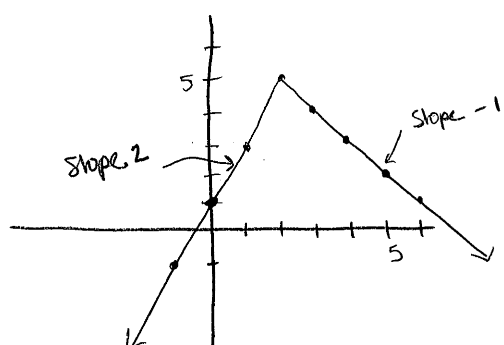
c) Give any sets of consecutive points for which the average rates of change of the function are approximately equal.

Avg. rate of change from B to C is approx. equal to avg. rate of change from D to E.

d) Sketch a tangent line to the graph somewhere <u>between</u> the points B and C such that the slope of the tangent line you draw is the same as the average rate of change of the function between B and C. (Do you think it would be possible to do this for ANY two points on a curve?)

Yes ... this will be a theorem later :)

14. Sketch the graph of a continuous function f with $\underbrace{f(0)=1}_{(0,1)}$ and $f'(x)=\begin{cases} 2 & \text{if } x<2 \\ -1 & \text{if } x>2 \end{cases}$ * See below



All values left of x=2 have a slope of 2 | All values right of x=2 have a slope of -1.